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**A SURVEY OF THE THEORY OF THE  
BOUNDEDNESS, STABILITY, AND  
ASYMPTOTIC BEHAVIOR OF SOLUTIONS  
OF LINEAR AND NON-LINEAR DIFFERENTIAL  
AND DIFFERENCE EQUATIONS**

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6 A SURVEY OF THE THEORY OF THE BOUNDEDNESS,  
STABILITY, AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF  
LINEAR AND NON-LINEAR DIFFERENTIAL AND DIFFERENCE  
EQUATIONS,

by

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The nature of solutions of differential and difference equations is of great interest to the applied mathematician, the physicist, and the engineer. In particular, numerous problems of vital concern to the Navy in connection with the self-excitation of oscillations in electronic and mechanical systems, the stability of moving bodies, the steering and turning of ships, etc. lead to non-linear differential equations. Thus far, results in the theory of these equations have been comparatively disorganized, and this survey should be useful in the unification and further development of the field. The Office of Naval Research is therefore pleased to make this report available in accordance with its statutory function of disseminating scientific information.

*Alan T. Waterman*

Alan T. Waterman,  
Chief Scientist

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## Introduction

The class of differential equations of the form

$$(1) \quad \frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n, t), \quad 1 \leq i \leq n,$$

where  $x_1, x_2, \dots, x_n$  and  $t$  are real variables has played a prominent part in the mathematical investigation of the physical world, and the present emphasis on non-linear processes indicates strongly that its role will not diminish in the future.

Combined with the intrinsic mathematical interest of systems of differential equations is the vast range of practical applications in such diverse fields as aerodynamics, astrophysics, and electronics. This combination has resulted in an enormous mass of scientific papers, written by specialists in their respective journals. Much of this is relatively inaccessible, and for that reason unknown even to experts in the same field, as the great amount of duplication of results shows.

Consequently, it was felt that it would be useful to mathematicians and natural scientists alike to have the known results on the behavior of solutions of (1) as  $t \rightarrow +\infty$  collected and correlated. It is hoped that such a survey will stimulate research on outstanding problems, and prevent duplication of what is already known.

In order to make the results of the survey as widely available as possible, no previous knowledge of differential equations has been assumed. Further, all nomenclature and notation is defined at time of use.

As central themes, three principal properties of solutions

were selected, boundedness, stability, and asymptotic behavior. Although the first two properties are qualitative, and the third quantitative, all three are closely interconnected, and it is not easy to separate results into categories pertaining to one or the other property.

The restriction to real differential equations has materially limited the scope of the results concerning asymptotic behavior. However, to have included the complex plane would have meant doubling the size of the survey and introducing many additional complicated concepts and methods.

Nor is there anything on differential-difference equations, infinite systems of differential equations, product integration, linear and non-linear Sturm-Liouville theory, and, more generally, the behavior of solutions as a function of a parameter, iteration, existence and uniqueness theory, topological methods, and many other fundamental parts of the theory of differential equations. However, in slight compensation, difference equations have been considered, (Chapter IV).

These topics have not been included either because of the existence of comprehensive texts treating the subject in question, as, for example, is the case for linear Sturm-Liouville theory, or topological methods, or because the subject would require a separate survey for an adequate coverage.

It is a pleasure to acknowledge the helpful criticism of Professor J.P. LaSalle of the University of Notre Dame and of Professor S. Lefschetz of Princeton University who read parts of

the text in manuscript, and to thank Mr. Newton Hawley for an excellent job of proof-reading. However, the responsibility for such errors, omissions, etc. as remain is solely that of the author.

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Princeton University

## CHAPTER I

### Boundedness, Stability, and Asymptotic Behavior of Solutions of Systems of Linear Differential Equations

#### §1. Introduction

In this chapter, we shall study the behavior as  $t \rightarrow +\infty$  of solutions of systems of linear differential equations of the form

$$(1) \quad \frac{dz_1}{dt} = \sum_{j=1}^n a_{1j}(t)z_j, \quad i = 1, 2, \dots, n.$$

Unless otherwise stated, the dependent variables  $z_k$  will be assumed to be real functions of the independent variable  $t$ , which runs through the interval  $[t_0, \infty]$ . The coefficients  $a_{1j}(t)$  will also be taken to be real functions of  $t$ , absolutely integrable over any finite interval.

The treatment of systems of the form (1) is materially simplified by the use of vector-matrix notations. Let  $z$  be an  $n$ -dimensional column vector with the components  $z_1, z_2, \dots, z_n$  and  $A(t)$  the  $n \times n$  matrix  $(a_{1j}(t))$ ,  $i, j = 1, 2, \dots, n$ . Then (1) can be written in the concise form

$$(2) \quad \frac{dz}{dt} = A(t)z.$$

The magnitude of the vector will be measured by the norm,

$$(3) \quad \|z\| = \sum_{k=1}^n |z_k|.$$

Similarly, the norm for a matrix is

$$(4) \quad \|A\| = \sum_{j=1}^n |a_{1j}(t)|.$$

It is readily verified that these norms satisfy the usual rules, namely,

- (5) (a)  $\|z\| = 0$ , if and only if  $z = 0$ , the null-vector;  
 (b)  $\|y+z\| \leq \|y\| + \|z\|$ ;  
 (c)  $\|cz\| = |c| \|z\|$ , for any real constant  $c$ ;  
 (d)  $\|A+B\| \leq \|A\| + \|B\|$ ;  
 (e)  $\|AB\| \leq \|A\| \|B\|$ ,  $\|Az\| \leq \|A\| \|z\|$ .

The symbol  $|A|$  will be used to represent the determinant  $|a_{ij}(t)|$ , associated with the matrix  $A$ .  $A$  will be said to be singular or non-singular, accordingly as  $|A| = 0$  or  $|A| \neq 0$ . If  $A$  is non-singular,  $A^{-1}$  exists.

In what follows, our attention will be focussed mainly upon systems, since any  $n$ -th order linear differential equation can be transformed into an  $n \times n$  system. Thus, if

$$(6) \quad \frac{d^n u}{dt^n} + p_1(t) \frac{d^{n-1} u}{dt^{n-1}} + \dots + p_n(t)u = 0,$$

one may introduce new variables

$$(7) \quad \begin{aligned} u &= u_1 \\ \frac{du}{dt} &= u_2 \\ &\vdots \\ \frac{du_{n-1}}{dt} &= u_n. \end{aligned}$$

Then the problem of solving (6) is equivalent to the problem of solving the system

$$(8) \quad \begin{aligned} \frac{du_1}{dt} &= u_2 \\ \frac{du_2}{dt} &= u_3 \\ &\vdots \\ \frac{du_{n-1}}{dt} &= u_n \\ \frac{du_n}{dt} &= -p_n u_1 - p_{n-1} u_2 \cdots -p_1 u_n. \end{aligned}$$

Three related properties of the solutions of (2) will be considered in this chapter: boundedness, stability and asymptotic behavior. We begin by making these terms precise.

Definition: A solution of (2),  $z$ , will be said to be bounded if  $\|z\|$  is bounded as  $t \rightarrow +\infty$ .

The definition of stability is more difficult, and necessarily so. Stability is associated with variation, and before a solution can be judged stable, a knowledge of what factors have changed is required.

Considering an equation of the type given in (2), it is seen that the solution is changed if the initial condition, the value of  $z$  at  $t = 0$ , is altered, or if the matrix  $A(t)$  is altered.

It is reasonable to suppose that in many instances the solution of the original equation and the solution of the altered

or perturbed system, will differ very slightly in properties if the changes in the initial conditions and  $A(t)$  are "small enough." Naturally, this last expression must be defined carefully. For the present we use it in an intuitive sense.

In addition to altering  $A$ , the form of (2) may be varied. In place of the linear system (2), the more general system

$$(9) \quad \frac{dz}{dt} = A(t)z + f(z,t),$$

where  $f(z,t)$  is a non-linear vector function of  $z$ , may be considered.

In this case, it is relevant to ask whether there is any relation between the behavior of the solutions of (2) and (9). This is equivalent to examining the validity of using (2) as a first approximation to (9). This problem will be discussed in Chapter II.

Once we have determined some of the ways in which an equation can be altered, we turn to the solutions and ask what properties of those solutions are of interest. Clearly, boundedness is one such property; another is the fact that  $\|z\| \rightarrow 0$  as  $t \rightarrow +\infty$ . If the solution is unbounded, determination of its magnitude as  $t \rightarrow +\infty$  is of interest, and we may compare it with elementary functions such as  $t^k$ ,  $e^{at}$ . Integrability properties are sometimes of importance. We may wish to know whether

$$(10) \quad \int_0^{\infty} \|z\| dt, \text{ or } \int_0^{\infty} \|z\|^2 dt,$$

is convergent.

(We shall use the notation

$$(11) \quad \int_0^{\infty} \|z\| dt < \infty$$

to signify the fact that the integral is convergent.)

We are now in a position to define a type of stability of particular importance in the sequel.

Definition: The solutions  $z$  of

$$(12) \quad \frac{dz}{dt} = f(z, t)$$

are said to be stable with regard to a property  $P$  under a variation  $V$  of the form of the equation which converts (12) into

$$(13) \quad \frac{dw}{dt} = g(w, t),$$

if every solution  $w$  of (13) also has this property  $P$ .

In the applications, where only one property is of importance, and only one type of variation is applied, it is convenient to say merely "the solutions are stable." Actually, we do this in Chapter II. However, it must be pointed out that "stable" and "stability" are sadly abused and overworked words, varying greatly from context to context. Perhaps the statement that describes the situation best is that there is no stability to the definition of the word stability.

We shall mean, as often done in analysis, by

$$(14) \quad f(t) \sim g(t),$$

read  $f(t)$  is asymptotic to  $g(t)$ , that

$$(15) \quad \lim_{t \rightarrow +\infty} \frac{f(t)}{g(t)} = 1.$$

If  $f$  and  $g$  are vectors, (14) means that (15) holds for each of the corresponding components.

Subsequently, this definition will be broadened.

We may now define the third of the three properties to be investigated.

Definition: By asymptotic behavior of  $z$  as  $t \rightarrow \infty$ , we mean the behavior of  $z$  or  $\|z\|$  as  $t \rightarrow +\infty$  as compared to elementary functions such as  $t^k e^{at}$ .

Thus, for example, we shall investigate conditions under which

$$(16) \quad z \sim e^{at} c,$$

where  $c$  is a constant vector.

If (16) is not true, a weaker condition

$$(17) \quad \lim_{t \rightarrow +\infty} \frac{\log \|z\|}{t} = a, \text{ or } \log \|z\| \sim at,$$

may still hold.

We shall first discuss with the simplest type of equation (2), the case where  $A$  is a constant matrix. Once these results have been obtained, we shall possess a unit with which to judge solutions of (2) for which  $A$  is not constant.

## §2. Linear Differential Equations with Constant Matrix

In this section, some results concerning the equation

$$(1) \quad \frac{dy}{dt} = Ay,$$

where  $A$  is constant, will be collected for future reference. For a complete discussion, we refer the reader to Lefschetz's monograph, [19]. We shall present here a rapid survey of the theory of (1).

Let us endeavor to obtain a solution of the form  $y = e^{\lambda t}c$ , where  $c$  is a constant vector. The following linear homogeneous set of equations for the components is obtained:

$$(2) \quad \lambda c = Ac.$$

The well-known necessary and sufficient condition that  $c$  be a non-trivial vector (not equal to the null-vector) satisfying (2) yields the determinantal relation

$$(3) \quad |A - \lambda I| = 0.$$

This equation is called the characteristic equation of the matrix  $A$ , and the  $n$  roots of (3) are called the characteristic roots. A root  $\lambda$  of (3) determines a solution  $c(\lambda)e^{\lambda t}$  of (1). If  $\lambda$  is complex, to obtain real solutions it is necessary to take real and imaginary parts of  $c(\lambda)e^{\lambda t}$ .

Since  $y = c(\lambda)e^{\lambda t}$ , the geometric location of the roots in the complex  $\lambda$ -plane determines the asymptotic behavior of the solutions as  $t \rightarrow \infty$ . As a consequence of this remark, in theory at least, the problem of the asymptotic behavior of solutions of (1)

is completely answered by the classical reduction theory of matrices. In practice, however, when the order of the matrix is large and when parameters appear in A, the problem is of great difficulty. Although there are several usable criteria for determining the geometrical location of the roots of (3), Routh, Hurwitz, etc., the numerical labor involved is still great.

Let C be a constant matrix, and make the change of variable  $y = Cw$  in (1). The following equation is obtained for w:

$$(4) \quad \frac{dw}{dt} = C^{-1}ACw$$

It is known from matrix theory that, if the characteristic roots of A are  $\lambda_1$ , C can be chosen so that

$$(5) \quad C^{-1}AC = \begin{pmatrix} L_1 & & & 0 \\ & L_2 & & \\ & & \ddots & \\ 0 & & & L_m \end{pmatrix}$$

where

$$(6) \quad L_k = \begin{pmatrix} \lambda_k & 0 & 0 & \dots \\ 1 & \lambda_k & 0 & 0 & \dots \\ & 1 & & & \\ & & \ddots & & \\ \vdots & & & \ddots & \\ \vdots & \dots & & 1 & \lambda_k \end{pmatrix}$$

and several  $L_k$  may contain the same  $\lambda_k$ . We shall call the  $L_k$  elementary factors. The elementary factor is said to be simple if

$$L_k = (\lambda_k).$$

From (5) it is easy to prove that the solutions of (4) have the following form, if  $\lambda_k$  is a characteristic root for which  $L_k$  is an  $m_k \times m_k$  matrix,

$$(7) \quad y^{(1)} = e^{\lambda_k t} \begin{pmatrix} 1 \\ t \\ \frac{t^2}{2!} \\ \vdots \\ \frac{t^{m_k}}{m_k!} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad y^{(2)} = e^{\lambda_k t} \begin{pmatrix} 0 \\ 1 \\ \frac{t}{1!} \\ \vdots \\ \frac{t^{m_k-1}}{(m_k-1)!} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad y^{(m_k)} = e^{\lambda_k t} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This is the general form. It is easy to see that in particular cases, if the elementary factors are simple, even if  $\lambda_k$  is a multiple root, the powers of  $t$  will not occur.

(If  $\lambda_k$  is complex, we take real and imaginary parts of the solutions.)

Since the solutions of (1) are linear combinations of the solutions (4), we may state:

Theorem 1. The necessary and sufficient condition that every non-trivial solution of (1)  $\rightarrow 0$  as  $t \rightarrow +\infty$  is that the characteristic roots of  $A$  have negative real parts. The necessary and sufficient condition that all solutions of (1) are bounded as  $t \rightarrow +\infty$  is that the characteristic roots have non-positive real parts, and that those with zero real parts are associated with simple elementary factors. If  $k, k \leq n$ , of the characteristic roots have positive real parts, there is a  $k$ -dimensional linear manifold of solutions for which  $\|y\| \rightarrow +\infty$  as  $t \rightarrow +\infty$ .

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### §3. Linear Differential Equations with Periodic Matrix

The basis for most results concerning periodic systems is the following representation theorem:

Theorem 2. Every solution of

$$(1) \quad \frac{dz}{dt} = A(t)z,$$

A(t) periodic with period T, has the form

$$(2) \quad z = P(t)y,$$

where

- (3) (a) P(t) is periodic with period T and non-singular;  
 (b) y is a solution of an equation  $dy/dt = By$ , where B is a constant matrix.

For the proof, consult Lefschetz [20].

The characteristic roots of B are called the characteristic exponents of A(t). The following result follows immediately:

Theorem 3. The necessary and sufficient condition that all solutions of (1) be bounded is that all solutions of  $dy/dt = By$  be bounded.

The determination of the characteristic exponents is a problem of great difficulty, and seems to be essentially of a transcendental, i.e., non-algebraic, nature. There are some simple criteria in the case of second-order linear differential equations, which we shall present in Chapter III, but otherwise very little is known.

#### §4. Linear Differential Equations with Almost-periodic Matrix

Periodic systems are a small sub-class of a much more general class of almost-periodic systems.

Definition: The matrix  $A(t)$  is said to be almost-periodic if all its elements are uniformly convergent trigonometric series of the form

$$(1) \quad a_{ij}(t) = \sum_{k=-\infty}^{\infty} c_{ijk} e^{i\lambda_k t}, \quad -\infty < t < \infty, \quad \lambda_k \text{ real.}$$

An almost periodic vector is defined analogously. Here we consider the full infinite  $t$ -interval  $-\infty < t < \infty$ .

If the  $\lambda_k$  are all integral multiples of a common number  $d$ ,  $A(t)$  is actually periodic. Let us exclude this case when we speak of an almost-periodic matrix. The theory of almost-periodic systems suffers from a lack of a representation theorem corresponding to Theorem 2. It is stated in Cameron, [7], that no such representation theorem can exist in general.

For a very interesting paper which may admit of extension, we refer the reader to Shtokalo. [38].

#### §5. Some General Theorems concerning Systems

In the succeeding sections we shall treat solutions of equations of the type

$$(1) \quad \frac{dz}{dt} = A(t)z + w,$$

where we suppose the behavior of the solution of

$$(2) \quad \frac{dy}{dt} = A(t)y$$

known. To link the behavior of the solutions of (1) and (2), we shall use Volterra-type integral equations, and express the solutions of (1) as solutions of a certain integral equation involving solutions of (2).

We shall consistently use the following notations. We denote by  $Y$  the matrix solution of

$$(3) \quad \frac{dY}{dt} = A(t)Y, \quad Y(0) = I, \text{ the identity matrix.}$$

For any solution  $z$  of (1),  $y$  will denote the solution of (2) with the same initial value,  $z(0) = y(0)$ .

Then

Lemma 1.  $z$  is given by

$$(4) \quad z = y + \int_0^t Y(t)Y^{-1}(t_1)w(t_1)dt_1.$$

If  $A(t)$  is a constant matrix,  $Y(t)Y^{-1}(t_1) = Y(t-t_1)$ , and

(4) becomes

$$(5) \quad z = y + \int_0^t Y(t-t_1)w(t_1)dt_1.$$

If  $w(t) = w(z, t)$ , a function of  $z$  and  $t$ ,  $z$  satisfies the Volterra-type integral equation

$$(6) \quad z = y + \int_0^t Y(t)Y^{-1}(t_1)w(z, t_1)dt_1.$$

These formulas may be verified by direct substitution.

After the differential equation has been converted into an integral equation, the following result is often very helpful:

Lemma 2. If  $u, v \geq 0$ , and  $c$  is a positive constant, and

$$(5) \quad u \leq c + \int_0^t v(t_1)u(t_1)dt_1,$$

then

$$(6) \quad u \leq ce^{\int_0^t v(t_1)dt_1}$$

This lemma seems to have been used first by Gronwall, [15], and subsequently by Bellman, [1], [2], [3], Caligo, [6], Guilano, [16], and Weyl, [43].

#### Part I - Theorems on Stability

##### §6. Equations of the Form $dz/dt = (A(t) + B(t))z$ .

In this section, we begin the discussion of the behavior of solutions of equations of the type

$$(1) \quad \frac{dz}{dt} = (A(t) + B(t))z,$$

where the properties of the solutions of

$$(2) \quad \frac{dy}{dt} = A(t)y$$

are assumed known. The magnitude of  $\|B(t)\|$  may be estimated in various ways. For example, we may require one of the following

- (3) (a)  $\|B(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$ ,  
 (b)  $\int_0^{\infty} \|B(t)\| dt < \infty$ ,  
 (c)  $\int_0^{\infty} \|B(t)\|^2 dt < \infty$ .

We will consider various cases of equations, corresponding to choosing  $A$  a constant matrix, a periodic matrix, and so on. We will identify each case in the heading by the title "right-hand side of the form...". where it is understood we mean the right-hand side of equation (1).

§7. Right-hand Side of the Form  $(A + B(t))z$ ,  $A$  Constant.

Let us consider first the case where  $A(t) = A$ , a constant matrix. The following important theorem is due to Hukuwara, [17], for the case of systems, and to Spath, [39], for the case of  $n$ -th order linear differential equations.

Theorem 4. All solutions of

$$(4) \quad \frac{dz}{dt} = (A + B(t))z$$

are bounded, provided that the following conditions hold

- (5) (a)  $A$  is constant,  
 (b) all solutions of  $dy/dt = Ay$  are bounded.  
 (c)  $\int_0^{\infty} \|B\| dt < \infty$ .

Proofs of this theorem have also been given by Bellman, [3], Cesari, [9], Caligo, [6], Levinson, [24], and Weyl, [43]. A related, weaker theorem was given by Dini, [14].

We give the proof since it is very brief, and illustrates a method which can be used in many problems of this type.

Using Lemma 1 any solution of (4) satisfies an integral equation of the form

$$(6) \quad z = y + \int_0^t Y(t - t_1)B(t_1)z(t_1)dt_1.$$

According to the hypothesis,  $\|y\|$  and  $\|Y\|$  are bounded.

Hence

$$(7) \quad \|z\| \leq c_1 + c_2 \int_0^t \|B(t_1)\| \|z(t_1)\| dt_1.$$

Applying Lemma 2, this yields

$$(8) \quad \|z\| \leq c_1 e^{c_2 \int_0^t \|B(t_1)\| dt_1} \leq c_1 e^{c_2 \int_0^\infty \|B(t_1)\| dt_1}$$

and so by (5c),  $\|z\|$  is bounded.

§8. Right-hand Side of the Form  $(A(t)+B(t))z$ , where  $A(t)$  is Periodic.

We may state the following intuitive "principle" which will be illustrated by many of the subsequent theorems: "Whatever boundedness and stability results are valid for a constant matrix  $A$ , are valid also for a periodic matrix  $A$ ".

This is a consequence of the representation of solutions of equations of the form

$$(1) \quad \frac{dy}{dt} = A(t)y,$$

$A(t)$  periodic, given above §3, Theorem 2. Using this representation, we shall prove the following

Theorem 5. All solutions of

$$(2) \quad \frac{dz}{dt} = (A(t) + B(t))z$$

are bounded, provided that the following conditions hold:

- (3) (a) A(t) is periodic  
 (b) all solutions of  $dy/dt = A(t)y$  are bounded  
 (c)  $\int_0^{\infty} \|B\| dt < \infty$ .

The method of proof illustrates a general method of treating equations with periodic matrix.

Let Y be the matrix solution of

$$(4) \quad \frac{dY}{dt} = A(t)Y, \quad Y(0) = I,$$

and y the solution of

$$(5) \quad \frac{dy}{dt} = A(t)y$$

with the same initial value as z. Then z satisfies the integral equation

$$(6) \quad z = y + \int_0^t Y(t)Y^{-1}(t_1)B(t_1)z(t_1)dt_1.$$

As a consequence of Theorem 2, the representation theorem,  $y = P(t)w$ ,  $Y(t) = P(t)W(t)$ , where

$$(7) \quad \frac{dw}{dt} = Bw, \quad \frac{dW}{dt} = BW$$

$B$  constant, and  $P(t)$  periodic and non-singular. Thus  $\|w\|$ ,  $\|W\|$  are bounded, as a consequence of the hypothesis. Furthermore

$$(8) \quad Y(t)Y^{-1}(t_1) = P(t)W(t)W^{-1}(t_1)P^{-1}(t_1) = P(t)W(t - t_1)P^{-1}(t_1).$$

Hence

$$(9) \quad \|z\| \leq \|y\| + \int_0^t \|P(t)\| \|W(t - t_1)\| \|P^{-1}(t_1)\| \\ \|B(t_1)\| \|z(t_1)\| dt_1 \\ \leq c_1 + c_2 \int_0^t \|B(t_1)\| \|z(t_1)\| dt_1.$$

The proof is concluded upon applying Lemma 2.

Once it is known that  $\|z\|$  is bounded, one may be interested in  $\lim_{t \rightarrow \infty} z$ , if it exists, or in the oscillatory behavior of  $z$  as  $t \rightarrow \infty$ , if the limit does not exist.

Theorem 6. Every solution of

$$(10) \quad \frac{dz}{dt} = (A(t) + B(t))z$$

approaches an almost-periodic vector, which may be the zero vector, as  $t \rightarrow \infty$ , provided that the following conditions hold:

- (11) (a)  $A(t)$  is either constant or periodic.  
 (b) all solutions of  $dy/dt = A(t)y$  are bounded,  
 (c)  $\int_{\infty}^{\infty} \|B\| dt < \infty.$

The result follows readily from the integral equation

- (6). The case where  $A(t)$  is constant has been treated by Levinson, [24].

Whether the above theorems remain true, or not, if we lighten the condition upon  $A(t)$ , and assume it only to be almost-periodic, is not known, and seems to present a difficult problem.

If more stringent conditions are imposed upon  $A(t)$ , the restriction upon  $B(t)$  can be lightened. Thus

Theorem 7. Every solution of

$$(12) \quad \frac{dz}{dt} = (A(t) + B(t))z$$

approaches zero as  $t \rightarrow +\infty$ , provided that the following conditions hold:

- (13) (a)  $A(t)$  is either constant or periodic.  
 (b) all solutions of  $dy/dt = A(t)y \rightarrow 0$  as  $t \rightarrow +\infty$ ,  
 (c)  $\|B\| \rightarrow 0$  as  $t \rightarrow +\infty$ , or, more generally,  $\|B\|$  is sufficiently small for  $t \geq t_0$ .

§9. Right-hand Side of the Form  $(A+B(t))z$  with  $B(t)$  of Bounded Variation in  $(t_0, \infty)$ .

As will be shown by an example in Chapter III, the condi-

tion  $\|B(t)\| \rightarrow 0$  is not sufficient to insure that the solution of

$$(1) \quad \frac{dz}{dt} = (A + B(t))z$$

are bounded if the solutions of

$$(2) \quad \frac{dy}{dt} = Ay$$

are bounded. We have seen above that  $\int_0^{\infty} \|B(t)\| dt < \infty$  is sufficient for this to be true. However, the integrability condition is not necessary, and sometimes too restrictive a condition. The following result, due to Cesari, [9], is a generalization of Theorem 4.

Theorem 8, All solutions of

$$(3) \quad \frac{dz}{dt} = (A + B(t) + C(t))z$$

are bounded, provided that the following conditions hold:

- (4) (a)  $A(t)$  is constant  
 (b) all solutions of  $dy/dt = Ay$  are bounded as  $t \rightarrow +\infty$   
 (c)  $\int_0^{\infty} \|dB(t)\| < \infty$   
 (d)  $\int_0^{\infty} \|C(t)\| dt < \infty$   
 (e) the characteristic roots of  $\|A + B(t) - \lambda I\| = 0$ , as functions of  $t$ , have non-positive real parts for  $t \geq t_0$ .

Condition 4(c) is to be interpreted to mean that every element of  $B(t)$  is of bounded variation in an interval  $[t_0, \infty)$ .

§10 Right-hand Side of the Form  $(A(t) + B(t))z$ , The General Case.

Let us now consider the general case where  $A(t)$  is neither constant nor periodic. This situation is decidedly more difficult to handle, due to the fact that the kernel  $Y(t)Y^{-1}(t_1)$  will not, in the general case, have any particular simple form, as it does when  $A(t)$  is constant or periodic. As seen from the proofs of Theorems 4 and 5, given the boundedness of solutions of

$$(1) \quad \frac{dy}{dt} = A(t)y,$$

and

$$(2) \quad \int_0^{\infty} \|B(t)\| dt < \infty,$$

the boundedness of the solution of

$$(3) \quad \frac{dz}{dt} = (A(t) + B(t))z,$$

is a consequence of the boundedness of  $\|Y(t)Y^{-1}(t_1)\|$ , for  $t, t_1 \geq t_0$ .

This is to be expected owing to the following result:

Theorem 9. The necessary and sufficient condition that all solutions of

$$(4) \quad \frac{dz}{dt} = A(t)z + f(t)$$

be bounded for every vector  $f(t)$  satisfying the condition

$$(5) \quad \int_0^{\infty} \|f(t)\| dt < \infty,$$

is that

$$(6) \quad \|Y(t)Y^{-1}(t_1)\| \leq c_1, \quad t, t_1 \leq t_0,$$

where  $c_1$  is independent of  $t$  and  $t_1$ .

If the solutions are to be bounded for every vector satisfying the condition

$$(7) \quad \|f(t)\| \leq c_2 < \infty, \quad t \geq t_0,$$

then the necessary and sufficient condition is that

$$(8) \quad \int_{t_0}^t \|Y(t)Y^{-1}(t_1)\| dt_1 \leq c_3 < \infty, \quad t \geq t_0,$$

where  $c_3$  is independent of  $t$ .

In its original form, the result is due to Perron, [33]; cf. also Caligo, [6]. For the proof we refer to Bellman, [45]. We shall refer to this theorem when discussing non-linear systems in Chapter II.

To obtain boundedness theorems in the general case, it is necessary to know some simple conditions satisfied by  $A(t)$  which will insure that  $\|Y(t)Y^{-1}(t_1)\| \leq c_1$ . In the previous cases, because of the functional equation  $Y(t)Y^{-1}(t_1) = Y(t - t_1)$ , valid when  $A$  is constant, boundedness of  $Y$  was sufficient. Since  $\|Y(t)Y^{-1}(t_1)\| \leq \|Y(t)\| \|Y^{-1}(t_1)\|$ , and by hypothesis,  $\|Y\|$  is bounded, it is sufficient to obtain conditions which will imply that

$\|Y^{-1}(t)\|$  is bounded. The elements of  $Y^{-1}(t)$  are cofactors of elements of  $Y(t)$  divided by  $|Y|$ . Hence, it is enough to have  $|Y|^{-1}$  bounded as  $t \rightarrow +\infty$ . Now

$$(9) \quad |Y| = e^{\int_0^t (\text{trace } A) dt},$$

(see Lefschetz, [21]). Hence we require

$$(10) \quad \lim_{t \rightarrow \infty} \int_0^t \text{trace } A dt > -\infty.$$

This condition is certainly satisfied if  $\text{trace } A = 0$ . This is perhaps the most important case, and includes the particular equation

$$(11) \quad \frac{d^2 u}{dt^2} + a(t)u = 0,$$

which we consider again in Chapter III.

Thus we have the following result:

Theorem 10. All solutions of

$$(12) \quad \frac{dz}{dt} = (A(t) + B(t))z$$

are bounded, provided that the following conditions hold:

- (13) (a) all solutions of  $dy/dt = A(t)y$  are bounded,  
 (b)  $\lim_{t \rightarrow \infty} \int_0^t (\text{trace } A) dt > -\infty$   
 (c)  $\int_0^{\infty} \|B\| dt < \infty.$

For proofs, see Bellman, [3], Caligo, [6], Wintner, [44].

§11. Right-hand Side of the Form  $A(t) = A + \lambda B(t)$ , A Constant,  
B(t) Periodic

Consider the equation

$$(1) \quad \frac{dz}{dt} = (A + \lambda B(t))z$$

where  $A$  is a constant matrix, and  $B(t)$  is periodic with period  $T$ . If all the solutions of  $dy/dt = Ay$  are bounded as  $t \rightarrow +\infty$ , all the solutions of (1) need not be bounded, even if  $\lambda$  is arbitrarily small, for, if the period of a solution of  $dy/dt = Ay$  coincides with  $T$ , we may have a "resonance" effect. However, it might be suspected that, if no "resonance" effect occurs, then all solutions of (1) will be bounded for  $|\lambda|$  sufficiently small. This is actually so, and we have the following results due to Cesari, [11].

Theorem 11. Consider the equation

$$(2) \quad \frac{dz}{dt} = Az + \lambda B(t)z$$

where  $A$  is a diagonal matrix

$$(3) \quad A = \begin{pmatrix} \epsilon_1^2 & & & 0 \\ & \epsilon_2^2 & & \\ & & \ddots & \\ 0 & & & \epsilon_n^2 \end{pmatrix}, \quad \epsilon_1 \neq \epsilon_j, \quad \epsilon_i > 0,$$

$B(t)$  is a symmetric matrix which is periodic, of period  $T$ , and even,  $B(t) = B(-t)$ , subject to the condition that

$$(4) \quad (a) \quad \int_0^T B(t) dt = 0$$

(b) Every element of  $B$  has an absolutely convergent Fourier series, and  $\lambda$  is a real parameter.

Let  $w = 2\pi/T$ . If

$$(5) \quad mw \neq \begin{cases} \epsilon_1 + \epsilon_j \\ \epsilon_1 - \epsilon_j \end{cases}, \quad 1, j = 1, 2, \dots, \quad m = 1, 2, \dots,$$

then there exists a number  $\lambda_0 > 0$  such that for  $\lambda < \lambda_0$ , the solutions of (2) are bounded.

For the case of 2nd-order systems, the result can be generalized.

Theorem 12. Consider the 2nd-order system

$$(6) \quad \frac{dz}{dt} = (A + \lambda B(t))z$$

where

- (7) (a)  $B(t)$  is periodic of period T,  
 (b)  $\int_0^T B(t)dt = 0$   
 (c) every element of  $B(t)$  has an absolutely convergent  
Fourier series  
 (d) the characteristic roots of A are complex conjugate  
 $\rho \pm i\epsilon$  with real part non-positive.

Then if

$$(8) \quad m \neq 2\epsilon, \quad m = 1, 2, \dots$$

there exists a number  $\lambda_0 > 0$ , such that for  $|\lambda| < \lambda_0$ , the  
solutions of (6) are bounded.

Cesari, [11], showed by an example that the result does not hold in general for systems of order greater than two.

### Part II - Theorems on Boundedness

#### §12. Limits of Solutions of Linear Differential Equations

In this section, we discuss some results due to Spath, [39], and, in their original form, to Perron, [29], [30]. Consider the differential equations,

$$(1) \quad \frac{d^n u}{dt^n} + a_{n-1}(t) \frac{d^{n-1} u}{dt^{n-1}} + \dots + a_0(t)u = \phi(t),$$

$$(1)' \quad \frac{d^n u}{dt^n} + a_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \dots + a_0 u = \phi(t),$$

where

$$(2) \quad \lim_{t \rightarrow \infty} a_k(t) = a_k, \quad \lim_{t \rightarrow \infty} \phi(t) = b.$$

The question arises as to whether

$$(3) \quad \lim_{t \rightarrow \infty} u, \quad \lim_{t \rightarrow \infty} \frac{d^k u}{dt^k}$$

exist.

This question is answered by the following two propositions:

Theorem 13. Consider equation (1) where the  $a_k(t)$  are constant  
and

$$(4) \quad \lim_{t \rightarrow \infty} \phi(t) = b$$

Then for any solution  $u$  which approaches a limit as  
 $t \rightarrow +\infty$ , we have

$$(5) \quad \lim_{t \rightarrow \infty} \frac{du}{dt} = \lim_{t \rightarrow \infty} \frac{d^2 u}{dt^2} = \dots = \lim_{t \rightarrow \infty} \frac{d^n u}{dt^n} = 0.$$

Solutions of this type exist if for every pure imaginary  
root,  $r = i\alpha$ , of the characteristic equation

$$(6) \quad r^n + a_{n-1}r^{n-1} + \dots + a_0 = 0$$

of multiplicity  $q$ , the integrals

$$(7) \int_0^{\infty} e^{-i\alpha t} \psi(t) dt, \int_0^{\infty} dt_1 \int_{t_1}^{\infty} e^{-i\alpha t} \psi(t) dt, \dots \int_0^{\infty} \int_{t_{k-1}}^{\infty} \dots \int_{t_1}^{\infty} e^{-i\alpha t} \psi(t) dt,$$

exist, where

$$(8) \quad \psi(t) = \phi(t) - b,$$

If  $a_0 \neq 0$ , all such solutions approach the same limit

$$(9) \quad \lim_{t \rightarrow \infty} u = b/a_0.$$

If there are no pure imaginary roots, there is always one solution which has a finite limit. If all the roots have negative real parts, all solutions have this property. If all the roots have positive real parts, there is only one such solution.

If  $g$  is the number of roots with negative real parts, then  $u(0), u'(0), \dots, u^{(g-1)}(0)$  may be arbitrarily prescribed and a solution with finite limit having these initial values will exist.

In the case of variable coefficients, we obtain the following result, also due to Spath, [39]:

Theorem 14. Consider the equation

$$(10) \quad \frac{d^n u}{dt^n} + a_{n-1}(t) \frac{d^{n-1} u}{dt^{n-1}} + \dots + a_0(t)u = \phi(t)$$

where

$$(11) \quad \lim_{t \rightarrow \infty} a_k(t) = a_k, \quad \lim_{t \rightarrow \infty} \phi(t) = b.$$

If some of the roots of the characteristic equation are pure imaginary, let the greatest multiplicity be  $l$  and assume that

$$(12) \quad \int_0^{\infty} t^{l-1} d(t) dt < \infty,$$

where

$$(13) \quad d(t) = \sum_{k=1}^n |a_k - a_k(t)| + |\phi(t) - b|.$$

Then equations (10) and (1') have simultaneously solutions which together with their first  $(n-1)$  derivatives are bounded.

To every bounded solution  $\bar{u}$  of (1), there exists exactly one solution of (10),  $u$ , with the property that

$$(14) \quad \frac{d^k u}{dt^k} = \frac{d^k \bar{u}}{dt^k} + o(1) \text{ as } t \rightarrow \infty, \quad k = 1, 2, \dots, g,$$

$$\left. \frac{d^k u}{dt^k} \right|_{t=t_0} = \left. \frac{d^k \bar{u}}{dt^k} \right|_{t=t_0}$$

where  $g$  is the number of roots of (5) with negative real parts.

### §13. Generalized Hooke's Law

Extending the result that the solutions of

$$(1) \quad \frac{d^2 u}{dt^2} + a^2 u = 0$$

are bounded, we can establish the following theorem:

Theorem 15. All solutions of

$$(2) \quad \frac{d^2z}{dt^2} = (A + B(t) + C(t))z$$

are bounded, provided that

(3) (a) A is a constant, symmetric, negative definite matrix,

(b) B is symmetric,

$$(c) \quad (1+c_1) \left| \sum_{i,j=1}^n b_{ij}(t)x_1x_j \right| \leq \left| \sum_{i,j=1}^n a_{ij}x_1x_j \right|,$$

$$t \geq t_0, \quad c_1 > 0,$$

$$(d) \quad \int_0^{\infty} \|B'(t)\| dt < \infty,$$

$$(e) \quad \int_0^{\infty} \|C(t)\| dt < \infty.$$

For the proof we refer to Bellman, [3].

#### §14. Some General Results on Boundedness.

Let us examine the connection between  $\|y\|$  and  $\|A\|$  as  $t \rightarrow \infty$ . If  $A$  is a constant matrix, it is known, cf. Lefschetz, [22], that the solution of

$$(1) \quad \frac{dy}{dt} = Ay, \quad y(0) = y_0,$$

may be written

$$(2) \quad y = e^{At} y_0,$$

where  $e^{At}$  represents the matrix given by the infinite series

$$(3) \quad 1 + At + \frac{A^2 t^2}{2!} + \dots + \frac{A^n t^n}{n!} + \dots$$

Thus, from (2)

$$(4) \quad \|y\| \leq \|e^{At}\| \|y_0\| \leq e^{\|A\|t} \|y_0\|.$$

If  $A$  is a variable matrix, (2) is no longer valid, but (4) is. Thus

**Theorem 16.** The solution of (1) satisfies the inequality

$$(5) \quad \|y\| \leq \|y_0\| e^{\int_0^t \|A(t_1)\| dt_1}$$

**Proof:** Converting (1) into the integral equation

$$(6) \quad y = y_0 + \int_0^t Ay dt,$$

we have

$$(7) \quad \|y\| \leq \|y_0\| + \int_0^t \|A\| \|y\| dt,$$

and (5) follows upon applying Lemma 2.

An analogous result is due to Kitamura, [18]. Define, for the moment only, the norms as

$$(8) \quad \|y\| = \left( \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}}, \quad p \geq 1,$$

$$\|A\| = \max_{\|x\| > 0} \frac{\|A(t)x\|}{\|x\|}.$$

Then

Theorem 17.

$$(9) \quad \|y(0)\| e^{-\int_0^t \|A\| dt} \leq \|y(t)\| \leq \|y(0)\| e^{\int_0^t \|A\| dt}$$

Kitamura obtains a similar, more complicated result for the equation

$$(10) \quad \frac{dy}{dt} = Ay + w.$$

Related results are due to Toyama, [42].

A consequence of Theorem 16 is the following result of Trjitzinsky, [41]:

Theorem 18. If

$$(11) \quad \frac{dy}{dt} = Ay,$$

and

$$(12) \quad \int_0^{\infty} \|A\| dt < \infty,$$

then

$$(13) \quad \lim_{t \rightarrow +\infty} y$$

exists.

Cf. also Bellman, [3], Wintner, [44].

Trjitzinsky, [41], also proved

Theorem 19. If  $A(t) = (a_{ij}(t))$  and

$$(14) \quad |a_{ij}(t)| \leq a(t),$$

then if  $Y(t) = (y_{ij}(t))$  is the solution of

$$(15) \quad \frac{dY}{dt} = AY, \quad Y(t_0) = I,$$

we have

$$(16) \quad |y_{ij}(t) - \delta_{ij}| \leq \frac{1}{n} \left( e^{n \int_{t_0}^t a(t_1) dt_1} - 1 \right), \quad t \geq t_0.$$

If

$$(17) \quad \int_{t_0}^{\infty} a(t) dt < \infty,$$

there is a solution of

$$(18) \quad \frac{dY}{dt} = AY$$

with the property that  $Y(+\infty) = I$ .

The following device is often useful. Write

$$(19) \quad \frac{dy}{dt} = Ay$$

in the explicit form

$$(20) \quad \frac{dy_i}{dt} = \sum_{j=1}^n a_{ij}(t)y_j, \quad i = 1, 2, \dots, n.$$

Then multiply the  $i$ -th equation by  $y_i$  and sum. This yields

$$(21) \quad \frac{1}{2} \frac{d}{dt} \sum_{i=1}^n y_i^2 = \sum_{i,j=1}^n (a_{ij}(t) + a_{ji}(t))y_i y_j.$$

Integrating between 0 and  $t$ , there results

$$(22) \quad \sum_{i=1}^n y_i^2 = \sum_{i=1}^n y_i^2(0) + 2 \int_0^t \left\{ \sum_{i,j=1}^n [a_{ij}(t_1) + a_{ji}(t_1)] y_i y_j \right\} dt$$

Hence

$$(23) \quad \sum_{i=1}^n y_i^2 \leq \sum_{i=1}^n y_i^2(0) + \int_0^t \left( \sum_{i,j=1}^n |a_{ij}(t_1) + a_{ji}(t_1)| \right) \left( \sum_{i=1}^n y_i^2 \right) dt_1.$$

Applying Lemma 2, we have the following result due to Butlewski, [5], cf. Rosenblatt, [37], for an application:

Theorem 20. If

$$(24) \quad \sum_{i,j=1}^n \int_0^{\infty} |a_{ij} + a_{ji}| dt < \infty,$$

all solutions of (20) are bounded.

#### §15. Transformation Theorems

It is sometimes of theoretical importance to know that an equation of the form

$$(1) \quad \frac{dy}{dt} = A(t)y, \quad A(t) \text{ continuous}$$

can be transformed, by means of a substitution,

$$(2) \quad y = B(t)z,$$

into an equation of the form

$$(3) \quad \frac{dz}{dt} = A^*(t)z,$$

where  $A^*(t)$  is diagonal or semi-diagonal. A matrix  $A$  is semi-diagonal if  $a_{ij} = 0$  for  $i > j$  or  $i < j$ .

The following results, due to Diliberto, [13], are refinements of results due to Perron, [31].

Theorem 21. Consider the equation (1). There exists an orthogonal matrix  $B(t)$ , such that, if  $z$  is given by (2), then  $A^*(t)$  in (3) is semi-diagonal.

Naturally, the construction of the matrix  $B$  depends upon the knowledge of the solutions of (1).

Theorem 22. Consider the equation (1). There exists a bounded non-singular matrix  $B(t)$ , such that, if  $z$  is given by (2), then  $A^*(t)$  in (3) is diagonal.

#### §16. Generalized Characteristic Roots

Consider the equation

$$(1) \quad \frac{dy}{dt} = A(t)y.$$

If  $A(t)$  is constant, the characteristic roots determine the magnitude of  $\|y\|$ . If  $A(t)$  is periodic, the magnitude is determined by the characteristic exponents. If  $A(t)$  is a general

matrix, we reverse this process and determine numbers, which we call characteristic numbers, in terms of the magnitude of  $\|y\|$ .

Definition:  $c$  is a characteristic number if

$$(2) \quad c = \overline{\lim}_{t \rightarrow \infty} \frac{\log \|y\|}{t} .$$

This concept was introduced by Liapounoff, [25], and developed by Cotton, [12] and Perron, [32]. Diliberto, [13], used his results given in the preceding section to obtain simpler proofs of some results of Liapounoff and Perron.

It can be shown that there are only a finite number of characteristic numbers. Multiplicity can be defined as follows. Let

$$(3) \quad c_1 < c_2 < \dots < c_m$$

be the characteristic numbers. Let  $e_1$  be the number of linearly independent solutions of (1) with  $c_1$  as characteristic number. Let  $e_1 + e_2$  be the number with  $c_2$ , and so on. Then  $e_1$  is the multiplicity of  $c_1$ ,  $e_2$  that of  $c_2$ , and so on. Furthermore,  $n = \sum_k e_k$ .

We shall give two results of Perron, [32].

Theorem 23.

$$(4) \quad \sum_{k=1}^n c_k \geq \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \operatorname{Re}(\operatorname{trace} A) dt.$$

Theorem 24. If  $c_1 \geq c_2 \dots \geq c_n$  are the characteristic numbers, where multiple characteristic numbers are written as many times as they occur, of

$$(5) \quad \frac{dy}{dt} = A(t)y$$

and  $d_1 \leq d_2 \leq \dots \leq d_n$  those of the adjoint equation

$$(6) \quad \frac{dx}{dt} = -xA(t),$$

then

$$(7) \quad c_k + d_k \geq 0, \quad k = 1, 2, \dots, n.$$

### Part 3 - Theorems on Asymptotic Behavior

#### §17. Asymptotic Series

Let us consider the behavior of a function  $f(t)$ , defined over the positive real axis, as  $t \rightarrow \infty$ . Suppose that

$$(1) \quad \lim_{t \rightarrow +\infty} f(t) = a_0.$$

If we are interested in the finer details of the behavior of  $f(t)$ , we must examine the difference  $f(t) - a_0$ . Suppose that this difference tends to zero like  $1/t$ . We then examine

$$(2) \quad \lim_{t \rightarrow +\infty} t(f(t) - a_0) = a_1,$$

assuming that this limit exists. Continuing in this way, we may form

$$(3) \quad t^2(f(t) - a_0 - \frac{a_1}{t})$$

and investigate its limit as  $t \rightarrow +\infty$ .

If  $f(t)$  has the property that constants  $a_0, a_1, a_2, \dots, a_n, \dots$ , exist having the property that

$$(4) \quad \lim_{t \rightarrow +\infty} (f(t) - a_0) = 0,$$

$$\lim_{t \rightarrow +\infty} t^n(f(t) - a_0 - \frac{a_1}{t} \dots - \frac{a_{n-1}}{t^{n-1}}) = a_n, \quad n = 1, 2, \dots,$$

$f(t)$  is said to have an asymptotic expansion, and we write

$$(5) \quad f(t) \sim \sum_{n=0}^{\infty} \frac{a_n}{t^n}.$$

Although the sequence of functions  $t^{-n}$  is most common, we might also consider asymptotic developments of the form

$$(6) \quad f(t) \sim a_0 + \sum_{k=1}^{\infty} a_k \psi_k(t),$$

where  $\{\psi_k(t)\}$  is an arbitrary sequence of functions which approach 0 as  $t \rightarrow \infty$ , and the  $a_k$  are defined by

$$(7) \quad \begin{aligned} a_0 &= \lim_{t \rightarrow +\infty} f(t) \\ a_n &= \lim_{t \rightarrow +\infty} \frac{[f(t) - a_0 + \sum_{k=1}^{n-1} a_k \psi_k(t)]}{\psi_n(t)}. \end{aligned}$$

It follows from (4) that

$$(8) \quad \left| f(t) - \sum_{n=0}^N \frac{a_n}{t^n} \right| < \frac{c_{N+1}}{t^{N+1}},$$

for  $t$  above a certain value.

From this, it is clear that the partial sums  $\sum_{n=0}^N a_n t^{-n}$  furnish approximations to  $f(t)$  for large values of  $t$ . However, this does not imply that the infinite series  $\sum_{n=0}^{\infty} a_n t^{-n}$  converges for any finite  $t$ , no matter how large, since the  $a_n$  may increase very rapidly. For example, it is easy to show, by repeated integration by parts, that for  $t > 0$ ,

$$(9) \quad e^t \int_t^{\infty} \frac{e^{-s}}{s} ds \sim \frac{1}{t} - \frac{1}{t^2} + \frac{2!}{t^3} - \frac{3!}{t^4} + \dots + \frac{(-1)^{n+1} n!}{t^{n+1}} + \dots$$

However, the infinite series on the right diverges for all finite  $t$ .

To make the most advantageous use of asymptotic series, it is necessary, since the series usually diverges in practice, to choose  $N$ , the order of the partial sum, so that the error term  $c_{N+1} t^{-(N+1)}$  in (8) above is a minimum. Thus  $N$  will be a function of  $t$ . Using the exact formula

$$(10) \quad e^t \int_t^{\infty} \frac{e^{-s}}{s} ds = \frac{1}{t} - \frac{1}{t^2} \dots + \frac{(-1)^N n!}{t^{N+1}} + (-1)^N n! e^t \int_t^{\infty} \frac{e^{-s}}{s^{N+1}} ds,$$

and the estimate

$$(11) \quad \left| n! e^t \int_t^{\infty} \frac{e^{-s}}{s^{n+1}} ds \right| < \frac{n!}{t^{n+1}},$$

it is seen that  $N$  should be chosen to be  $[t]$  or  $[t]-1$  if  $t$  is non-integral, and  $t-1$  if  $t$  is an integer.

The first important asymptotic series was that of Stirling for  $\log \Gamma(z)$ . This may be obtained by integrating by parts the integral in the following formula

$$(12) \quad \log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \int_t^{\infty} \left( \frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{1}{2} \right) \frac{e^{-zx}}{x} dx,$$

where the real part of  $z$  is positive.

The Laplace transform

$$(13) \quad F(t) = \int_0^{\infty} f(s)e^{-st} ds$$

is the source of many asymptotic developments obtained by means of repeated integration by parts.

Let us now discuss the application of asymptotic series to the differential equations. The first systematic use of asymptotic series is due to Poincaré, [35], although a specialized theory was considered independently by Stieltjes, [40]. Poincaré applied his theory to differential equations, for which it was invented, while Stieltjes was interested in the moment-problem and continued fractions.

That the application to differential equations is successful is due to the fact that asymptotic series possess many of the properties of convergent series. We state these properties as a lemma:

Lemma 4. If

$$(14) \quad f(t) \sim \sum_{n=0}^{\infty} a_n t^{-n}, \quad g(t) \sim \sum_{n=0}^{\infty} b_n t^{-n},$$

then

$$(15) \quad f(t) + g(t) \sim \sum_{n=0}^{\infty} (a_n + b_n) t^{-n},$$

$$(16) \quad f(t)g(t) \sim \sum_{n=0}^{\infty} c_n t^{-n}, \quad c_n = \sum_{k=0}^n a_k b_{n-k},$$

$$(17) \quad \text{if } f'(t) \sim \sum_{n=2}^{\infty} d_n t^{-n}, \quad \text{then } d_n = -(n-1)a_{n-1},$$

$$(18) \quad \text{if } a_0 = a_1 = 0, \quad \int_t^{\infty} f(t) dt \sim \sum_{n=2}^{\infty} \frac{a_n t^{-(n-1)}}{(n-1)}.$$

From these properties, it follows that if  $f, f', \dots, f^{(n)}$ , have asymptotic developments, then  $P(f, f', \dots, f^{(n)})$ , where  $P$  is a polynomial in its variables, also has an asymptotic development, obtained in the obvious manner. Perhaps the main problem in the application of asymptotic series to differential equations is the following: "Given an asymptotic series, divergent for all finite  $t$ , which formally satisfies the equation  $P(f, f', \dots, f^{(n)}) = 0$ , under what conditions is the series an asymptotic expansion of a solution of this differential equation"

For a discussion of this problem we refer to Borel, [4], Remoundos, [36].

The general problem of the asymptotic development of solutions of equations of the form

$$(19) \quad \frac{dy}{dt} = A(t)y$$

where the elements of  $A(t)$  are polynomials in  $t$  was first attacked by Poincaré, [35]. His researches were continued by Horn, Birkhoff, and Trjitzinsky. The latter gave an essentially complete solution to the problem as far as asymptotic representation was concerned, and also considered the representation of the solution by means of convergent factorial series.

For a complete discussion of the methods and results of the above-cited authors it is necessary to enter the complex plane. Since we have agreed to limit our discussion to the behavior of solutions on the real axis, we shall refer the reader to Horn, Trjitzinsky, [41], where extensive references are given.

In this chapter, we shall consider linear equations and give a partial answer to the question above. In Chapter II, non-linear equations will be considered.

The concept of asymptotic development can be generalized. If there exist functions  $a_n(t)$ ,  $\psi(t)$ ,  $\phi(t)$ , where  $\psi(t) \rightarrow \infty$ , and the  $a_n(t)$  are bounded, with the property that

$$f(t) = \phi(t) \sum_{n=0}^N \frac{a_n(t)}{\psi(t)^n} + \frac{\phi(t)\epsilon_N(t)}{\psi(t)^N},$$

where  $\epsilon_N(t) \rightarrow 0$  as  $t \rightarrow \infty$ , we write

$$f(t) \sim \phi(t) \sum_{n=0}^{\infty} \frac{a_n(t)}{\psi(t)^n}$$

This definition coincides with the former definition for the case  $\phi(t) = 1$ ,  $\psi(t) = t$ ,  $a_n(t) = a_n$ .

§18. Asymptotic Development of Solutions of Linear Differential Equations

In this section, we consider the linear differential equation

$$(1) \quad \frac{dz}{dt} = A(t)z,$$

where the elements of  $A(t)$  have asymptotic developments of the form (5) of the previous section. Let us further assume that the elements of  $\frac{dA}{dt}$  also possess asymptotic developments.

The following result is due to Hukuwara, [17]:

Theorem 25. Consider the equation (1) where

(2) (a) every element  $a_{ij}(t)$  of  $A(t)$  has an asymptotic development

$$a_{ij}(t) \sim \sum_{k=0}^{\infty} \frac{a_{ij}^{(k)}}{t^k},$$

- (b) the characteristic roots  $\lambda_1, \lambda_2, \dots, \lambda_n$  of  $A^0 = (a_{ij}^{(0)})$  are all distinct.

Then there exist  $n$  linearly independent solutions of (1),  $z^{(1)}, z^{(2)}, \dots, z^{(n)}$ , having the asymptotic expansions

$$(3) \quad z^{(i)} \sim e^{\lambda_i t} t^{\rho_i} \sum_{k=0}^{\infty} \frac{a_k^{(i)}}{t^k},$$

where the  $a_k^{(i)}$  are constant, possibly complex, vectors.

If the  $\lambda_i$  are complex, it is necessary to take real and imaginary parts of the  $z^{(i)}$  to obtain real solutions. Since complex  $\lambda_i$  occur in conjugate pairs, no additional solutions are obtained in this way.

#### §19. Asymptotic Development of Solutions of Linear Differential Equations - Periodic Coefficients

Consider the equation

$$(1) \quad \frac{dz}{dt} = A(t)z$$

where every element  $a_{ij}(t)$  of  $A(t)$  has the generalized asymptotic development

$$(2) \quad a_{ij}(t) \sim \sum_{k=0}^{\infty} \frac{c_{ij}^{(k)}(t)}{t^k},$$

where  $c_{ij}^{(k)}(t)$  is continuous and periodic of period  $T$ .

For this case Carleman, [8] proved the following theorem analogous to Theorem 25.

Theorem 26. Consider the equation (1), where

- (3) (a) every element of  $A(t)$  has an asymptotic development of the form (2),  
 (b) the characteristic exponents,  $\alpha_1, \alpha_2, \dots, \alpha_n$ , associated with the matrix  $C^0 = (c_{ij}^{(0)}(t))$  are all distinct,  
 (c)  $\alpha_i - \alpha_j \neq \frac{2k\pi}{T}$ ,  $i \neq j$ ,  $k = 0, 1, 2, \dots$ .

Then there exists a fundamental system of solutions of (1),  $z^{(1)}, z^{(2)}, \dots, z^{(n)}$ , having asymptotic developments

$$(4) \quad z^{(k)} \sim e^{\alpha_k t} t^{\rho_k} \sum_{l=0}^{\infty} \frac{d_l^{(k)}(t)}{t^l},$$

where the  $d_l^{(k)}(t)$  are continuous periodic vectors of period  $T$ .

§ 20.  $A(t)$  is "almost-constant"

We shall say that the matrix  $A(t)$  is "almost-constant" whenever there is a constant matrix  $A$  such that

$$(1) \quad \lim_{t \rightarrow \infty} A(t) = A.$$

The problem is to relate the behavior of  $\|z\|$  as  $t \rightarrow \infty$  where  $z$  satisfies

$$(2) \quad \frac{dz}{dt} = A(t)z$$

to the behavior of  $y$ , where  $y$  satisfies

$$(3) \quad \frac{dy}{dt} = Ay.$$

This problem was first considered by Poincaré, [34], for  $n$ -th order linear differential equations, under certain restrictive conditions. These restrictions were removed by Perron, [27], [28]. His results were in turn improved by Lettenmeyer, [23], and Hukuwara, [17].

The following result is due to Lettenmeyer, [23]:

Theorem 27. Consider the equation (2) where

$$(4) \quad (a) \quad \lim_{t \rightarrow \infty} A(t) = A$$

(b) the characteristic roots of  $A$ ,  $\lambda_1, \lambda_2, \dots, \lambda_n$ , are all distinct.

Then there exist  $n$  linearly independent solutions of (2),

$z^{(1)}, z^{(2)}, \dots, z^{(n)}$ , such that

$$(5) \quad \lim_{t \rightarrow \infty} \frac{\log \|z^{(k)}\|}{t} = R(\lambda_k),$$

Here, as usual in analysis,  $R(\lambda_k)$  stands for real part of  $\lambda_k$ .

Specializing this result to  $n$ -th order linear differential equations, we obtain

Theorem 28. Consider the differential equation

$$(6) \quad u^{(n)} + p_1(t) u^{(n-1)} + \dots + p_n(t) u = 0$$

where

$$(7) \quad (a) \quad \lim_{t \rightarrow \infty} p_k(t) = p_k$$

(b) the roots  $r_1, r_2, \dots, r_n$  of  $\sum_{k=0}^n p_k r^{n-k} = 0$  are all real and distinct.

Then there exist  $n$  linearly independent solutions  $u^{(1)}, u^{(2)}, \dots, u^{(n)}$ , such that

$$(8) \quad \lim_{t \rightarrow \infty} \frac{\frac{du}{dt}(k)}{u(k)} = r_k.$$

Further results, which are not of such simple nature, may

be obtained in the case of multiple roots. We refer to Perron, [28], Lettenmeyer, [23].

For the case of real, simple characteristic roots, we may also obtain more precise results for systems. Thus, we have Bellman, [3]:

Theorem 29. Consider the equation

$$(9) \quad \frac{dz}{dt} = (A + B(t))z$$

where

- (10) (a) A is a constant matrix, all of whose characteristic roots  $\lambda_1, \lambda_2, \dots, \lambda_n$  are real and distinct  
 (b)  $\|B\| \rightarrow 0$  as  $t \rightarrow +\infty$ .

Then corresponding to any characteristic root  $\lambda_k$ , there is a solution  $z^{(k)}$  satisfying the inequalities

$$(11) \quad c_2 \exp(\lambda_k t - d_2 \int_{t_0}^t \|B\| dt) \leq \|z^{(k)}\| \\ \leq c_1 \exp(\lambda_k t + d_1 \int_{t_0}^t \|B\| dt), \quad t \geq t_0, \quad c_2 \neq 0.$$

If we put a further restriction on B, we can obtain asymptotic results.

Theorem 30. If in equation (9),

- (12) (a) A is a constant matrix with real, distinct characteristic roots,  $\lambda_1, \lambda_2, \dots, \lambda_n$ ,  
 (b)  $\int_{\infty}^{\infty} \|B\| dt < \infty.$

Then corresponding to any characteristic root  $\lambda_k$ , there is a solution  $z^{(k)}$  having the property that

$$(13) \quad \lim_{t \rightarrow \infty} z^{(k)} e^{-\lambda_k t} = c_k,$$

where  $c_k$  is a non-zero constant vector.

Specializing this result to the case of an n-th order linear differential equation, we obtain

Theorem 31. Consider the equation

$$(14) \quad u^{(n)} + p_1(t)u^{(n-1)} + \dots + p_n(t)u = 0$$

where

- (15) (a)  $\lim_{t \rightarrow \infty} P_K(t) = P_K,$   
 (b)  $\int_{\infty}^{\infty} |P_K - P_K(t)| dt < \infty, \quad K = 1, 2, \dots, n,$   
 (c) the roots  $r_1, r_2, \dots, r_n$  of  $\sum_{K=0}^n P_K r^{n-K} = 0$  are real and distinct.

Then there exist  $n$  linearly independent solutions  
 $u_1, u_2, \dots, u_n$ , with the property that

$$(1c) \quad \lim_{t \rightarrow \infty} u_k e^{-r_k t} = 1$$

$$\lim_{t \rightarrow \infty} \frac{d}{dt} u_k e^{-r_k t} = r_k$$

Theorems 30 and 31 are due in their original form to Dini, [14], and Love, [20], and in their final form to Dunkel, [46], where the case of multiple characteristic roots is also considered.

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CHAPTER II  
NON-LINEAR DIFFERENTIAL EQUATIONS

§1. Introduction

In this chapter we shall study systems of non-linear differential equations of the form

$$(1) \quad \frac{dz}{dt} = F(z,t),$$

where  $F(z,t)$  is a non-linear vector function of  $z$ . Equations of this type are of great importance in celestial mechanics, which accounted for the original interest in these problems. Recently, they have also become of great importance in the study of mechanical and electrical circuits as the need for greater precision and explanation of new phenomena has forced physicists and engineers to consider non-linear equations.

We shall consider the case where

$$(2) \quad F(z,t) = A(t)z + f(z,t),$$

and  $\|f(z,t)\|$  is small compared to  $\|z\|$  as  $\|z\| \rightarrow 0$ . A simple example of this would be where every component of  $f(z,t)$  is a power series in  $z_1, z_2, \dots, z_n$ , beginning with terms of the second degree.

Since it is usually not possible to solve (1) explicitly in terms of known functions, it is necessary to develop some other means of determining the behavior of the solution. Of particular

interest in many physical problems is the behavior as  $t \rightarrow +\infty$ .

There are several different approaches to this question. Perhaps the one that is most intuitive is that which proceeds as follows. Let  $A(t)$  be a constant matrix, with characteristic roots having negative real parts. Then all solutions of

$$(3) \quad \frac{dy}{dt} = Ay$$

are bounded and  $\rightarrow 0$  as  $t \rightarrow +\infty$ . Hence, we may expect that, if  $\|z(0)\|$  is sufficiently small and  $\|f(z,t)\| / \|z\|$  is also sufficiently small for  $\|z\|$  small, the behavior of the solutions of

$$(4) \quad \frac{dz}{dt} = Az + f(z,t)$$

will parallel that of the solutions of (3). We shall see that this expectation is valid. The problem stated precisely is to examine the validity of the first approximation -- equation (3) -- to equation (4). This problem was first investigated by Poincaré, and then extensively and intensively treated by Liapounoff in a classic memoir. However, the problem has not been completely solved, and many questions remain unanswered.

Of at least as great importance as the behavior of solutions as  $t \rightarrow +\infty$ , and intimately connected with it, is the question of the existence of periodic solutions of (1). This question is now of great importance in connection with electronics. However,

we shall not consider the topic here, since it is not possible to present it simply and in few pages, and since it is treated at length in the easily available monograph of Lefschetz, [17]. Nevertheless, we shall discuss some results concerning the nature of the solutions of (1) when  $F(z,t)$  is periodic in  $t$ , or is derived in a special manner from a periodic function.

Finally the asymptotic behavior of solutions of (1) will be considered, and, more specifically, the behavior of solutions of equations of the form

$$(5) \quad P\left(u, \frac{du}{dt}, t\right) = 0,$$

$P$  a polynomial, as  $t \rightarrow +\infty$ . This problem has relevance to the problem of the behavior of the solutions of second-order linear differential equations.

## §2. Methods

Although we shall not give any proofs, some discussion of the methods used in obtaining the results given below seems in order. Foremost is the method of integral equations. The link between the solutions of (3) and (4) of §1 is furnished by the following lemma:

Lemma 1. Let  $y$  be the solution of

$$(1) \quad \frac{dy}{dt} = A(t)y$$

with the same initial value as  $z$ . Let  $Y(t)$  satisfy

$$(2) \quad \frac{dY}{dt} = A(t)Y, \quad Y(0) = I.$$

Then every solution of

$$(3) \quad \frac{dz}{dt} = A(t)z + f(z,t), \quad z(0) = z_0,$$

satisfies the integral equation

$$(4) \quad z = y + \int_0^t Y(t)Y^{-1}(t_1)f(z, t_1)dt_1.$$

As always in the theory of differential equations, the **advantage** of using an integral equation in place of the original differential equation lies in the smoothing properties of the integral operator as contrasted with the harsh behavior of the derivative. Thus, for example, if two functions are close, integration preserves this closeness, while differentiation may not even be applicable to the functions if they are merely continuous.

Once the integral equation has been obtained, an immediate technique is the method of successive approximations. Form the sequence

$$(5) \quad \begin{aligned} z_0 &= y \\ z_{n+1} &= y + \int_0^t Y(t)Y^{-1}(t_1)f(z_n, t_1)dt_1, \quad n = 0, 1, \dots \end{aligned}$$

It is now not difficult to show, under various assumptions concerning the nature of  $A(t)$  and  $f(z,t)$ , that the sequence  $\{z_n\}$  converges to a solution  $z$  of (4) and thence (3), having many of the properties of  $y$ .

This furnishes a constructive proof of the existence of various classes of solutions. If we are merely interested in proving the existence of certain solutions, we may regard (4) as an equation of the type

$$(6) \quad z = T(z),$$

where  $T(z)$  is a non-linear operator defined by the right-hand side of (4). The existence of a solution of (4) now depends upon the existence of a function  $z$  satisfying (6). Considering  $z$  as a point in an abstract space, we require a "fixed-point" of the transformation  $T(z)$ . Such a fixed-point will exist for a large class of operators, to which  $T(z)$  belongs, as was first shown by Birkhoff and Kellogg, [3]. For an application of this method to the problems of this chapter, see Hukuwara, [14], Bellman, [1].

Another method of approach is by means of difference equations. We approximate to the differential quotient by quotients

$$(7) \quad \frac{z(t+h) - z(t)}{h} = A(t)z + f(z,t), \quad t = 0, h, 2h, \dots$$

This method will be discussed in detail in Chapter IV.

A method of an entirely different type depends upon the connection between differential systems of the type

$$(8) \quad \frac{dz_1}{F_1(z)} = \frac{dz_2}{F_2(z)} = \dots = \frac{dz_n}{F_n(z)}$$

and linear partial differential equations of the form

$$(9) \quad \sum_{k=1}^n F_k(z) \frac{\partial u}{\partial z_k} = 0$$

This method is discussed by Liapounoff, [19], and used to treat some cases not amenable to the previous methods. Since Liapounoff's memoir has been reprinted [19], and is readily available, we will not enter into a discussion of this method.

### §3. Stability

We shall use the word stable in the following sense:

Definition: A solution  $z$  of

$$(1) \quad \frac{dz}{dt} = F(z,t)$$

is said to be stable if every other solution  $w$ , for which the difference  $\|w(t_0) - z(t_0)\|$  is sufficiently small, (non-zero), remains within a certain neighborhood of  $z$  for  $t \geq t_0$ ; that is, if

$\delta$  is sufficiently small and

$$(2) \quad \|w(t_0) - z(t_0)\| \leq \delta,$$

then

$$(3) \quad \|w(t) - z(t)\| \leq \epsilon(\delta), \quad t \geq t_0.$$

In many important cases;

$$(4) \quad \lim_{t \rightarrow +\infty} \|w(t) - z(t)\| = 0.$$

To illustrate this concept, let us consider the equation

$$(5) \quad \frac{dz}{dt} = Az + f(z,t)$$

where  $A$  is constant and  $f(0,t) = 0$ . Here it is clear that  $z = 0$  is a solution. The question arises as to whether it is a stable solution. Under the condition that all the characteristic roots of  $A$  have negative real parts, it is, as we shall discuss below.

For further discussion of stability we refer to Fejer, [10], Horn, [13], Lefschetz, [17], Levi-Civita, [18], [ ], Liapounoff, [19].

§4. The equation  $dz/dt = Az + f(z,t)$ ,  $A$  constant

In this section we shall begin answering the question raised in the previous section concerning the stability of  $z = 0$  as a solution of

$$(1) \quad \frac{dz}{dt} = Az + f(z,t),$$

where  $A$  is a constant matrix.

The following results are due to Perron, [28]. Other less general results were obtained by Liapounoff, [19], Poincaré, [30], Bohl, [4], Cotton, [5].

Theorem 1: If  $f(z,t)$  is a continuous function of  $z$  for  $\|z\| \leq c$ ,  
and

- (2) (a)  $\frac{\|f(z,t)\|}{\|z\|} \rightarrow 0$  as  $\|z\| \rightarrow 0$ , uniformly in  $t$ ,  
(b) all the characteristic roots of  $A$  have negative real parts,

then

- (3) (a)  $z = 0$  is a stable solution of (1),  
(b) every solution  $z$  for which  $\|z(0)\|$  is sufficiently small, has the property that,  $\|z\| \rightarrow 0$  as  $t \rightarrow \infty$ .

We shall use the term instable to indicate that  $z = 0$  is not stable. We then have the following result:

Theorem 2. If  $f(z,t)$  is a continuous function of  $z$  for  $\|z\| \leq c_1$ ,  
and

- (4) (a)  $\frac{\|f(z,t)\|}{\|z\|} \rightarrow 0$  as  $\|z\| \rightarrow 0$ , uniformly in  $t$ ,  
(b) at least one characteristic root of  $A$  has positive real part,

then  $z = 0$  is instable.

There still remains the possibility of conditional stability, that is, if not every solution for which  $\|z(0)\|$  is small remains in a bounded neighborhood of the origin, at least some submanifold of solutions remains in a bounded neighborhood. We expect this to occur if some of the characteristic roots of  $A$  have negative real parts. That this is true is shown by

**Theorem 3.** If

- (5) (a)  $f(z,t)$  is a continuous function of  $\|z\|$  for  $\|z\| \leq c_1$ ,  
 (b)  $\|f(z_1,t) - f(z_2,t)\| \leq \epsilon \|z_1 - z_2\|$ , whenever  
 $\|z_1\| \leq \delta(\epsilon)$ ,  $\|z_2\| \leq \delta(\epsilon)$ ,  
 (c)  $k$  of the characteristic roots of  $A$  have negative real parts, and  $n-k$  have positive real parts,

then there exists a  $k$ -dimensional manifold of solutions of (1) which  $\rightarrow 0$  as  $t \rightarrow \infty$ , and for which  $\|z(0)\| \leq \delta$  implies that  $\|z(t)\| \leq \epsilon(\delta)$  for  $t \geq 0$ , and an  $(n-k)$ -dimensional manifold for which this last condition is not valid.

If some of the  $(n-k)$  characteristic roots with non-negative real parts have zero real parts, we can only say that there exist at least a  $k$ -dimensional manifold of solution with the above property. We shall discuss later the case where characteristic roots with zero real parts occur, and we shall see that it is quite difficult to ascertain the general behavior.

The condition  $\|f(z,t)\| / \|z\| \rightarrow 0$  as  $\|z\| \rightarrow 0$  is clearly satisfied if every component of  $f(z,t)$  is a power series

in the components of  $z$ , lacking constant and first-degree terms. This was the case discussed by Liapounoff and Poincaré. Naturally, far more precise results can be obtained in this case, and we will refer to this topic again below.

It should also be mentioned that this condition 4(a) can be weakened to  $\|f(z,t)\| \leq \epsilon \|z\|$  whenever  $\|z\| \leq \delta(\epsilon)$ .

The above theorems illustrates the fact that essentially the behavior of the solutions of (1) as  $t \rightarrow +\infty$  is determined by the behavior of the solutions of

$$(5) \quad \frac{dy}{dt} = Ay$$

as  $t \rightarrow +\infty$ .

#### §5. Continuation

In the previous section, the cases where the characteristic roots had either positive or negative real parts were treated. In this section we shall treat the case where the characteristic roots may have zero real parts.

Theorem 4. If

- (1) (a) all the solutions of  $dy/dt = Ay$  are bounded as  $t \rightarrow \infty$   
 (b)  $\|f(z,t)\| \leq \epsilon \|z\| f(t)$ , whenever  $\|z\| \leq \delta(\epsilon)$ ,  
where  $\int_0^{\infty} f(t)dt < \infty$ ,

then  $z = 0$  is a stable solution of

$$(2) \quad \frac{dz}{dt} = Az + f(z,t).$$

For a proof, with more restrictive conditions imposed upon  $f(z,t)$ , we refer to Bellman, [1]. The theorem in its present form is due to Levinson.\* It may also be shown that  $z$  approaches an almost-periodic vector as  $t \rightarrow +\infty$ .

#### §6. $A(t)$ a periodic matrix

As shown in the first chapter, results valid for  $A$  constant carry over to  $A$  periodic, as far as boundedness is concerned. Thus we have the following theorem:

Theorem 5. If

- (1) (a) all solutions of  $dy/dt = A(t)y \rightarrow 0$  as  $t \rightarrow +\infty$ ,  
 $A(t)$  periodic,  
 (b)  $\|f(z,t)\|/\|z\| \rightarrow 0$  as  $\|z\| \rightarrow 0$ ,

then  $z = 0$  is a stable solution of

$$(2) \quad \frac{dz}{dt} = A(t)z + f(z,t)$$

and, in addition, every solution for which  $\|z(0)\|$  is sufficiently small  $\rightarrow 0$  as  $t \rightarrow +\infty$ .

This theorem is of some interest in the theory of variational equations. Suppose that we have a system

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\*Written communication to the author.

$$(3) \quad \frac{dz}{dt} = f(z,t)$$

which we know possesses a periodic solution. In some cases, it is of importance to know the behavior of "nearby" solutions. It may even be true that any solution which comes sufficiently close to this periodic solution must approach this solution more and more closely as  $t \rightarrow +\infty$ .

To decide this, let  $p$  be the given periodic solution and set  $z = p + w$ . Assuming that the components of  $f(z,t)$  are analytic functions of the components of  $z$ , we have

$$(4) \quad \frac{dz}{dt} = \frac{dp}{dt} + \frac{dw}{dt} = f(p+w,t) = f(p,t) + J\left(\frac{f(p,t)}{p}\right) w + \dots,$$

$J\left(\frac{f(p,t)}{p}\right)$  is the Jacobian matrix  $\left(\frac{\partial f_i}{\partial p_j}\right)$ , or

$$(5) \quad \frac{dw}{dt} = J\left(\frac{f(p,t)}{p}\right) w + \dots$$

which is an equation of the type treated in this section, provided  $f(p,t)$  is a periodic function of  $t$ . This will certainly be so, if, as is usually the case,  $f(z,t)$  does not contain  $t$  explicitly.

To decide whether or not  $w \rightarrow 0$  as  $t \rightarrow +\infty$ , whence  $z \rightarrow p$ , it is thus sufficient to consider the first approximation

$$(6) \quad \frac{dw}{dt} = J\left(\frac{f(p,t)}{p}\right) w.$$

A more general problem is furnished by (3) when the solutions are almost-periodic. Whether or not the analogue of Theorem 5 is true is not known.

#### §7. Generalizations

The previous results can be generalized to include equations of the form

$$(1) \quad \frac{dz}{dt} = Az + f(z, \frac{dz}{dt}, t).$$

We have the following result, cf. Bellman, [1].

#### Theorem 6. If

- (2) (a)  $k, 1 \leq k \leq n$ , of the characteristic roots of  $A$  have negative real part
- (b)  $\|f(u, v, t) - f(u_1, v_1, t)\| \leq \epsilon [\|u - u_1\| + \|v - v_1\|]$ ,  
whenever  $\|u\|, \|u_1\|, \|v\|, \|v_1\|$  are all less than  
 $\delta(\epsilon)$ ,

then (1) has a  $k$ -dimensional manifold of solutions which  $\rightarrow 0$  as  $t \rightarrow \infty$ .

#### §8. $A(t)$ is a variable matrix

If  $A(t)$  is a variable matrix which is not periodic, it is necessary to impose some more stringent restrictions upon  $A$  to conclude from the boundedness of the solutions of

$$(1) \quad \frac{dy}{dt} = A(t)y,$$

the boundedness of the solutions of

$$(2) \quad \frac{dz}{dt} = A(t)z + f(z,t).$$

This is due to the fact that the functional equation  $Y(t)Y^{-1}(t_1) = Y(t-t_1)$  is true only if  $A$  is a constant matrix, and if  $A$  is not constant or periodic there is generally no simple way of treating  $Y(t)Y^{-1}(t_1)$  as a function of two variables,  $t$  and  $t_1$ . For a discussion of cases where  $Y(t)Y^{-1}(t_1)$  may be handled easily, we refer to Trjitzinsky, [3].

As discussed in the first chapter, if  $\|Y(t)\|$  is bounded as  $t \rightarrow \infty$ , and

$$(3) \quad \lim_{t \rightarrow +\infty} \int_0^t \text{trace } A \, dt > -\infty,$$

then  $\|Y^{-1}(t)\|$  will also be bounded. The most important example of this condition being satisfied is when  $\text{trace } A = 0$ .

We have the following result, Bellman, [1].

**Theorem 7.** If

(4) (a) all solutions of  $dy/dt = A(t)y$  are bounded as  $t \rightarrow \infty$

(b)  $\lim_{t \rightarrow +\infty} \int_0^t \text{trace } A \, dt > -\infty$

(c)  $\|f(z,t)\| \leq \varepsilon f(t) \|z\|$ , whenever  $\|z\| \leq \delta(\varepsilon)$ ,  
and  $\int_0^{\infty} f(t) dt < \infty$ ,

then  $z = 0$  is a stable solution of (2).

§9. A general condition

The following theorem emphasizes the fact that the boundedness of the solutions of non-linear differential equations depends essentially upon the boundedness of the solutions of first approximation.

Theorem 8. The necessary and sufficient condition that the solution of

$$(1) \quad \frac{dz}{dt} = A(t)z + \phi(z, t)$$

be bounded for every vector function  $\phi(z, t)$  which is bounded for  $t \geq 0$  and  $z$  arbitrary is that the linear system

$$(2) \quad \frac{dz}{dt} = A(t)z + \phi(t)$$

possess only bounded solutions for all  $\phi(t)$  satisfying the condition

$$(3) \quad \|\phi(t)\| \leq c_1, \quad t \geq 0.$$

If this condition is satisfied and

$$(4) \quad \|\phi(z, t)\| \leq \|z\|$$

for  $\|z\| \leq \delta(\epsilon)$ , then  $z = 0$  is a stable solution of (1), and  $\|z\| \rightarrow 0$  as  $t \rightarrow \infty$ .

If (2) possesses only bounded solutions for all  $\phi(t)$  satisfying the condition

$$(5) \quad \int_0^{\infty} \|\phi(t)\| dt < \infty.$$

then  $z = 0$  is a stable solution of (1) if

$$(6) \quad \|\phi(z,t)\| \leq \|z\| f(t), \quad \int_0^{\infty} f(t) dt < \infty,$$

for  $\|z\| \leq \delta(\epsilon)$ .

We refer to Bellman, [1], for the proof.

#### §10. A Counter-Example

At this point, it might be suspected that a result of the following type would be true:

"If all solutions of

$$(1) \quad \frac{dy}{dt} = A(t)y$$

$\rightarrow 0$  as  $t \rightarrow +\infty$ , then all solutions of

$$(2) \quad \frac{dz}{dt} = A(t)z + f(z)$$

do likewise provided that  $\|f(z)\| / \|z\| \rightarrow 0$  as  $\|z\| \rightarrow 0$ , and  $\|z(0)\|$  is sufficiently small."

That this is not true is shown by the following example of Perron, [28]:

$$(3) \quad \frac{dy_1}{dt} = -ay_1, \quad (a > 0),$$

$$\frac{dy_2}{dt} = (\sin \log t + \cos \log t - 2a) y_2.$$

The general integral is

$$(4) \quad \begin{aligned} y_1 &= c_1 e^{-at} \\ y_2 &= c_2 e^{(\sin \log t - 2a)t} \end{aligned}$$

which  $\rightarrow 0$  as  $t \rightarrow +\infty$ , for  $a > 1/2$ .

Now consider the non-linear system

$$(5) \quad \begin{aligned} \frac{dz_1}{dt} &= -az_1, \\ \frac{dz_2}{dt} &= (\sin \log t - \cos \log t - 2a)z_2 + z_1^2 \end{aligned}$$

The general integral is

$$(6) \quad \begin{aligned} z_1 &= c_1 e^{-at} \\ z_2 &= e^{(\sin \log t - 2a)t} \left( c_2 + c_1^2 \int_0^t e^{-t} \sin \log t \, dt \right) \end{aligned}$$

It may be shown easily that if  $1 < 2a < 1 + e^{-\pi/2}$ ,  $z_2 \rightarrow 0$  as  $t \rightarrow +\infty$  only if  $c_1 = 0$ . Thus the condition  $\|z(0)\|$  sufficiently small does not entail  $\|z\| \rightarrow 0$  as  $t \rightarrow +\infty$ .

#### §11. One characteristic root zero

We now turn to the difficult question of deciding stability when some of the characteristic roots of  $A$  are zero. We begin with the simplest case where one characteristic root is zero. In this case, contrary to the results above, it is not sufficient to consider the linear terms alone, and the nature of the non-linear terms is critical. For this reason the results are quite complicated,

and we refer to the memoir of Liapounoff, [19], and later papers of Malkin, [25], [26], [ ], [ ], [ ].

By means of a transformation to polar coordinates, the case where there are two complex conjugate roots with zero real parts can be reduced to the case of one characteristic root zero. This is also discussed by Liapounoff, [19].

### §12. Asymptotic behavior of solutions

The question of the asymptotic behavior of solutions of

$$(1) \quad \frac{dz}{dt} = Az + f(z,t)$$

leads to the study of the geometric nature of the solutions considered as curves in  $z$ -space.

Since a detailed discussion of the two-dimensional case is given in Iefschetz, [17], we shall not enter into the subject here. We also refer to papers by Martin, [27], Petrowsky, [29], Poincaré, [30], [ ], [ ]. Weyl, [34], and Yosida, [35].

For a general treatment of asymptotic solutions of non-linear differential equations we refer to Trjitzinsky, [31], [32].

### §13. Transformation of equations

In many cases, it is possible to find transformations of the variables which reduce

$$(1) \quad \frac{dz}{dt} = Az + f(z,t)$$

to various canonical forms, which are easier to treat.

We refer the interested reader to Dulac, [9], Lefschetz, [17], MacMillan, [21], [ ] .

#### §14. All characteristic roots zero

While it is difficult to obtain results for the case where some of the characteristic roots are zero, Maillet, [22], [23], [24], has treated some cases, important in applications, where all the characteristic roots are zero. His chief result, [24], is the following:

Theorem 9. Consider the systems

$$(1) \quad \frac{dy}{dt} = \phi(y)$$

$$(2) \quad \frac{dz}{dt} = \phi(z) + \psi(z),$$

where all the components of  $\phi(y)$  are homogeneous polynomials in  $y_1, y_2, \dots, y_n$  of degree  $p > 1$ ,  $p$  odd, and  $\psi(y)$  has terms of higher order in  $y_1, y_2, \dots, y_n$ .

Let  $b_1$  be the coefficient of  $y_1^p$  in  $\phi_1(y)$ . Then the necessary and sufficient condition that the solutions of (1)  $\rightarrow 0$  as  $t \rightarrow \infty$  is that  $b_1 < 0$ .

The solutions of (2)  $\rightarrow 0$  as  $t \rightarrow +\infty$  if the solutions

of (1)  $\rightarrow 0$  as  $t \rightarrow +\infty$ , and the solutions of (2) then have the form

$$(3) \quad z_1 = z_2 = \dots = z_n = 0, \quad z_j = (c_j + \varepsilon_j(t)) (a+t)^{\frac{1}{1-p}},$$

$$j = 1+1, 1+2, \dots,$$

$c_j \neq 0$ , where  $\lim_{t \rightarrow \infty} \varepsilon_j(t) = 0$ , and

$$(4) \quad y_1 = y_2 = \dots = y_1 = 0, \quad y_j = c_j (a+t)^{\frac{1}{1-p}}, \quad j = 1+1, 1+2, \dots$$

is a solution of (1).

Theorem 10. If  $p$  is even, the above conditions, plus the additional restrictions that  $\phi_i(y) = y_i X_i(y)$ , where  $X_i$  is homogeneous of degree  $p-1$ , and that the initial values are positive and sufficiently small, are necessary and sufficient that the solutions of (2)  $\rightarrow 0$  as  $t \rightarrow +\infty$  when the same is true of the solutions of (1).

#### §15. Solutions in trigonometric form

Although we shall not discuss the general theory of solutions of equations of the type

$$(1) \quad \frac{dz}{dt} = Az + f(z,t)$$

where  $f(z,t)$  contains trigonometric terms and the characteristic roots of  $A$  have zero real part, since there are expositions of this theory in Lefschetz, [17], and Kryloff and Bogoliuboff, [16],

the following result due to Bohl, [5], deserves mention.

Theorem 11. Consider the equation

$$(2) \quad \frac{dz}{dt} = Az + f(z,t)$$

where

- (3) (a) no characteristic root of A has zero real part,  
 (b)  $\|f(z,t)\|/\|z\|$ ,  $\|f_z(z,t)\|^*$  are sufficiently small for  $\|z\|$  small, uniformly in t  
 (c)  $f(z,t)$  is obtained by the substitutions  $u_K = z/\alpha_K$  in the vector  $w(u_1, u_2, \dots, u_m, t)$ , which is continuous for all  $u_K$  and periodic with respect to the  $u_K$  of period one, where  $\alpha_K \neq 0$ , and no relation of the form

$$\sum_{K=1}^m c_K / \alpha_K = 0,$$

$c_K$  integral exists.

Then, provided that c is sufficiently small, there exists one and only one solution of (2), defined for all t, such that  $\|z\| < c$  for all t. The components of this solution are trigonometric series of the form

\*  $\|f_z(z,t)\|$  denotes the maximum of the norms of  $f_{z_1}(z,t)$ , where the subscript signifies partial differentiation with respect to that variable.

$$(4) \quad z_K = \sum_{i=1}^{\infty} z_{iK}, \quad K = 1, 2, \dots, n,$$

uniformly convergent for all t, where each term  $z_{iK}$  is a polynomial in  $\cos 2\pi t/\alpha_L, \sin 2\pi t/\alpha_L, L = 1, 2, \dots, M.$

If  $w(u_1, u_2, \dots, u_m, t)$  has continuous partial derivatives with respect to  $u_K$  and  $t$  up to the  $2n$ -th order, then each  $z_{iK}$  can be represented by a uniformly and absolutely convergent series of the form

$$(5) \quad \sum a_{v_1 \dots v_m} \cos 2\pi(v_1 t/\alpha_1 + d_1) \dots \cos 2\pi(v_m t/\alpha_m + d_m)$$

where

$$v_i = 0, 1, 2, \dots, \text{ and } d_i = 0 \text{ or } \frac{1}{4}.$$

§16. The magnitude of solutions of non-linear differential equations

Let  $P(x_1, x_2, \dots, x_n)$  be a polynomial in the  $n$  variables  $x_1, x_2, \dots, x_n$ , and consider the algebraic differential equation

$$(1) \quad P(t, u, \frac{du}{dt}, \dots, \frac{d^{n-1}u}{dt^{n-1}}) = 0.$$

The problem of estimating the behavior of real continuous solutions of (1) as  $t \rightarrow +\infty$  was first attacked by Borel, [6], His results were, in turn, refined by Lindelöf, [20]. The result of Lindelöf is

Theorem 12. If  $u$  is a real continuous solution of

$$(2) \quad P(t, u, \frac{du}{dt}) = 0,$$

for  $t \geq t_0$ , where  $P$  is a polynomial, then if  $a(t)$  is any real function with the property that as  $t \rightarrow +\infty$

$$(3) \quad \frac{a(t)}{t^n} \rightarrow +\infty, \quad n = 1, 2, \dots,$$

we have

$$(4) \quad |u| \leq e^{\int_{t_0}^t a(t) dt}, \quad t \geq t_0.$$

If  $P(t, u, \frac{du}{dt})$  is of degree  $m$  in  $t$ , then there exists a constant  $C$ , such that

$$(5) \quad |u| \leq e^{Ct^{m+1}}, \quad t \geq t_0.$$

The first part of the theorem with  $a(t) = e^t$  is due to Borel.

It was shown by Vijayaraghavan, [33], that the analogous result with  $\exp(e^t)$  in place of  $e^t$  is not true for equations of the form  $P(t, u, \frac{du}{dt}, \frac{d^2u}{dt^2}) = 0$ .

For equations of the form

$$(6) \quad \frac{du}{dt} = \frac{P(u, t)}{Q(u, t)}$$

where  $P$  and  $Q$  are polynomials, much more precise results are available. Equations of this type were investigated for  $t$  complex

by Boutroux. [7], and by Hardy. [12], for  $t$  real. We shall present the results of Hardy.

Theorem 13. Any solution of

$$(7) \quad \frac{du}{dt} = \frac{P(u,t)}{Q(u,t)}$$

continuous for  $t \geq t_0$  is ultimately monotonic, together with its derivatives, and satisfies one of the relations

$$(8) \quad u \sim c_1 t^{c_2 p(t)}, \quad u \sim (t^{c_3} \log t)^{1/c_4},$$

where  $p(t)$  is a polynomial,  $c_3, c_4$  are integers.

Any solution of  $P(t, u, u') = 0$  satisfies either

$$(9) \quad |u| \leq c_2 t^{c_1}$$

or

$$(10) \quad u = e^{c_3 t^{c_4} (1 + \varepsilon(t))} \quad \varepsilon(t) \rightarrow 0,$$

as  $t \rightarrow \infty$ .

All solutions of the latter class are monotonic, together with their derivatives.

Hardy gives some further results on the behavior of solutions of

$$(11) \quad \left(\frac{du}{dt}\right)^n = \frac{P(u,t)}{Q(u,t)}$$

Fowler, [11], considered the equation

$$(12) \quad \frac{d^2 u}{dt^2} = \frac{P(u,t)}{Q(u,t)},$$

and proved

Theorem 14. If  $u$  is a continuous solution of (12) with continuous first and second derivatives, for  $t \geq t_0$ , then there exist constants  $c_1, c_2, c_3, c_4$  such that either

$$(13) \quad |u| \leq c_1 t^{c_2}$$

or

$$(14) \quad u = e^{c_3 t^{c_4}} (1 + \epsilon(t)),$$

where  $\epsilon(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

#### §17. Asymptotic Behavior of Solutions of a Special Class of Equations

In the first chapter we discussed the behavior of solutions of equations of the form

$$(1) \quad \sum_{k=0}^n a_k(t) u^{(k)}(t) = 0,$$

where  $a_k(t) \rightarrow a_k$ , a constant, as  $t \rightarrow +\infty$ .

These results have been considerably extended by Koksma, [15], who considered the more general equation

$$(2) \quad \sum_{k=0}^n a_k(t) u^{(k)}(t) = r(t, u, du/dt, \dots, \frac{d^{(n)}(u)}{dt^n}).$$

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## CHAPTER III

### ON THE SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

#### §1. Introduction.

The second order linear differential equation,

$$(1) \quad \frac{d}{dt} \left( K(t) \frac{du}{dt} \right) + L(t)u = 0,$$

is of great importance in mathematical physics. It arises naturally from boundary-value problems in the theory of partial differential equations, occurs in the simpler form

$$(2) \quad \frac{d^2u}{dt^2} + a(t)u = 0,$$

as an analytical expression of Newton's laws of motion in dynamics, and is, again in the form (2), a fundamental equation in quantum mechanics (one-dimensional).

The mathematically trivial, but physically important, equation where  $a(t) = \pm a^2$  can be completely integrated. In one case, all solutions are periodic, with period  $2\pi/a$ , have an infinite number of zeroes in the interval  $(0, \infty)$ , and do not tend to any limit as  $t \rightarrow \infty$ ; in the other case, all solutions are ultimately monotonic, thus have at most a finite number of zeroes in  $(0, \infty)$ , and  $\rightarrow 0$  or  $\pm \infty$  as  $t \rightarrow \infty$ .

This chapter will be devoted principally to investigating which of these properties are retained by solutions of (2), and under

what conditions. Specifically, we are interested in the boundedness or unboundedness of the solutions as  $t \rightarrow +\infty$ , the number of zeroes of solutions in the interval  $[t, +\infty]$ , and the asymptotic behavior of the solution. As before, all terms such as bounded, positive and so forth, shall refer only to the interval  $[t_0, +\infty]$ ,  $t_0 > 0$ , unless specifically stated otherwise.

§2. A Preliminary Transformation.

Henceforth we shall consider equations of the form

$$(1) \quad \frac{d^2 u}{dt^2} + a(t)u = 0,$$

since the substitution

$$(2) \quad u = v e^{-\frac{1}{2} \int^t p dt}$$

eliminates the middle term in

$$(3) \quad \frac{d^2 u}{dt^2} + p(t) \frac{du}{dt} + q(t)u = 0,$$

and yields the following equation for  $v$

$$(4) \quad \frac{d^2 v}{dt^2} + (q(t) - \frac{1}{2} dp/dt - p^2/4)v = 0.$$

The problem of eliminating the two middle terms in

$$(5) \quad \frac{d^3 u}{dt^3} + p(t) \frac{d^2 u}{dt^2} + q(t) \frac{du}{dt} + r(t)u = 0$$

is much more difficult but has been solved by Forsyth.

By means of the transformation above, all theorems stated in terms of equation (1) have analogues for (3).

### §3. The Boundedness of Solutions of $u'' + a(t)u = 0$ .

Let us now discuss sufficient conditions for the boundedness of all solutions of

$$(1) \quad \frac{d^2 u}{dt^2} + a(t)u = 0.$$

Several important criteria may be derived from the following simple inequality:

Theorem 1. of  $a(t) > 0$  for all  $t \geq 0$ , then any solution of (1) satisfies the inequality

$$(2) \quad u^2 < \frac{c_1}{a(t)} \exp \left( \int_0^t \frac{|a'(t)|}{a(t)} dt \right)$$

Proof: We have

$$(3) \quad u' u'' + a(t)u u' = 0.$$

Integrating between 0 and  $t$ , and integrating the second term by parts,

$$(4) \quad \frac{u'^2}{2} + a(t)\frac{u^2}{2} = a(0)\frac{u^2(0)}{2} + \frac{u'(0)^2}{2} + \int_0^t a'(t)\frac{u^2}{2} dt$$

Thus,

$$(5) \quad \frac{a(t)u^2}{2} < c_2 + \int_0^t \frac{|a'(t)|}{a(t)} a(t)\frac{u^2}{2} dt.$$

Applying lemma 2 of Chapter I,

$$(6) \quad a(t)\frac{u^2}{2} < c_2 \exp\left(\int_0^t \frac{a'(t)}{a(t)} dt\right).$$

cf. Bellman, [5], Caccioppoli, [15].

As consequences of Theorem 1, we have the following

results:

Theorem 2. All solutions of (1) are bounded if  $a(t) > 0$ ,  $a'(t) > 0$ ,  $t > t_0$ .

Theorem 3. All solutions of (1) are bounded if  $a(t) > c_1 > 0$ , and is of bounded variation in some interval  $[t_0, \infty]$ . In particular, all solutions of

$$(11) \quad \frac{d^2u}{dt^2} + (a^2 + \phi(t))u = 0$$

are bounded if

$$(8) \quad (a) \quad a^2 + \phi(t) > b^2 > 0, \quad t > t_0 > 0$$

$$(b) \quad \int_{t_0}^{\infty} d\phi < \infty.$$

Theorem 2 is due to Biernacki, [7], Osgood, [41], and Theorem 3 to Caccioppoli, [15], and Wiman, [54], independently. For the above unified presentation, see Bellman, [5].

Equation (1) corresponds to the system

$$(9) \quad \begin{aligned} u_1' &= u_2 \\ u_2' &= -a(t)u_1 \end{aligned}$$

The matrix

$$(10) \quad A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}$$

has trace zero, and thus a corollary of Theorem 10 of Chapter I is the following result:

Theorem 4. All solutions of

$$(11) \quad \frac{d^2 u}{dt^2} + (a(t) + b(t))u = 0$$

are bounded, provided

(12) (a) All solutions of  $u'' + a(t)u = 0$  are bounded,

$$(b) \int^{\infty} |b(t)| dt < \infty .$$

In turn, a corollary of this result is

Theorem 5. If

$$(13) \int^{\infty} |a(t)| dt < \infty ,$$

all solutions of (10) cannot be bounded.

It follows from Hukuwara's result, Theorem 4 of Chapter I, that all solutions of (7) are bounded if

$$(14) \int^{\infty} |\phi(t)| dt < \infty .$$

Combining Theorems 3 and 4, we obtain the more general result:

Theorem 6. All solutions of

$$(15) \frac{d^2u}{dt^2} + (a^2 + \phi(t) + \psi(t))u = 0$$

are bounded if

$$(16) \quad (a) \quad a^2 + \phi(t) > b^2 > 0 \quad t > t_0,$$

$$(b) \quad \int_{t_0}^{\infty} d\phi < \infty,$$

$$(c) \quad \int_{t_0}^{\infty} |w| dt < \infty.$$

Butlewski, [12], generalized Theorem 2 and proved

Theorem 7. All solutions of

$$(17) \quad \frac{d}{dt} \left( \theta(t) \frac{du}{dt} \right) + a(t)u = 0$$

are bounded, provided,

$$(18) \quad (a) \quad a(t) > 0, \theta(t) > 0, \frac{d}{dt} (a(t) \theta(t)) > 0, \quad t > t_0.$$

All solutions of

$$(19) \quad \frac{d}{dt} \left( \theta(t) \frac{du}{dt} \right) + \sum_{i=0}^m a_{2i+1}(t) u^{2i+1} = 0$$

are bounded, provided,

$$(20) \quad (a) \quad \theta(t) > 0, a_{2i+1}(t) > 0, \frac{d}{dt} (\theta(t) a_{2i+1}(t)) > 0, \quad t \geq t_0$$

The following results due to Murray, [40], are also appropriate here:

Theorem 8. The equation

$$(21) \quad \frac{d^2u}{dt^2} - \phi(t)u = 0$$

can have no solution bounded for  $-\infty < t < +\infty$  if  $\phi(t) > \alpha > 0$ , for  $-\infty < t < \infty$ .\*

If

$$(22) \quad (a) \quad 0 < b^2 < \phi(t) < a^2, \quad -\infty < t < \infty,$$

$$(b) \quad |\psi(t)| < c_1, \quad -\infty < t < \infty,$$

there is one and only one solution of

$$(23) \quad \frac{d^2u}{dt^2} - \phi(t)u = \psi(t),$$

which is bounded for  $-\infty < t < \infty$ .

This result can be extended to equations of the form

$$(24) \quad \frac{d}{dt} (k(t) \frac{du}{dt}) - d(t)u = 0,$$

provided  $0 < a < k(t) < b$ , and continuous.

\* The trivial solution  $u = 0$  is, as usual, ignored.

§4. Counter-Examples.

The question arises as to whether or not the hypotheses of the previous theorems were too restrictive. Intuitively, one might expect that the condition  $\phi(t) \rightarrow 0$  as  $t \rightarrow +\infty$  would be sufficient to ensure the boundedness of all solutions of

$$(1) \quad \frac{d^2 u}{dt^2} + (a^2 + \phi(t))u = 0.$$

This was actually stated by Fatou, [18], but shown to be false by means of a counter-example by Perron, [42], cf. Caccioppoli, [15], Wintner, [55]. We shall give a specific counter-example due in slightly different form to Wintner, [55], and then a general method of constructing counter-examples.

Theorem 9. The equation

$$(2) \quad \frac{d^2 u}{dt^2} + \left(1 + 4 \frac{\sin 2t}{t}\right)u = 0$$

has unbounded solutions.

Another immediate way of realizing that the condition  $\phi(t) \rightarrow 0$  as  $t \rightarrow +\infty$  might not be sufficient to ensure boundedness of the solutions of (1) is to note that the theory of the Mathieu equation shows that the condition

$$(3) \quad 0 < a_1^2 < a^2 + \phi(t) < a_2^2$$

is not sufficient to ensure boundedness of solutions of (1). We shall discuss this condition later.

Wintner's general result is the following:

Theorem 10. If

$$(4) \quad g(t) \rightarrow 0, \quad g'(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

then

$$(5) \quad u = \exp \left( \int_0^t g(s) \cos s \, ds \right) \cos t$$

is a solution of

$$(6) \quad \frac{d^2 u}{dt^2} + (1 + \phi(t))u = 0,$$

where  $\phi \rightarrow 0$  as  $t \rightarrow +\infty$

Choosing  $g(s) = \cos s/s$ , we see that the conditions  $g(t) \rightarrow 0$ ,  $g'(t) \rightarrow 0$  are satisfied, while  $u$  is unbounded.

Levinson, [35], gave a counter-example of a different type.

Theorem 11. Consider the equation

$$(7) \quad \frac{d^2 u}{dt^2} + (a^2 + \phi(t))u = 0$$

Let  $\alpha(t)$  be a monotone increasing function such that  
 $\alpha'(t) = o(1)$  as  $t \rightarrow +\infty$ . Then there exists a  $\phi(t)$  such that  
for large  $t$

$$(8) \quad \int_0^t \phi(t) dt < \alpha(t),$$

and (7) has a solution satisfying

$$(9) \quad \overline{\lim}_{t \rightarrow \infty} \frac{\log |u|}{\alpha(t)} \gg \frac{1}{\pi}$$

§5. L<sup>-</sup>-stability.

On the above results we have considered the property of boundedness of solutions of

$$(1) \quad \frac{d^2 u}{dt^2} + a(t)u = 0.$$

We now consider the property of solutions belonging to  $L^p(0, \infty)$ ,  $p > 1$ . A function  $u$  is said to belong to  $L^p(0, \infty)$  if

$$(2) \quad \int_0^{\infty} |u|^p dt < \infty.$$

Generalizing results of Weyl, [5], and Carleman, [17], Bellman, [6],\* proved

Theorem 12. If all solutions of

$$(3) \quad \frac{d^2 u}{dt^2} + a(t)u = 0$$

\* Theorem 4 of Bellman, [6], is incorrect.

belong to  $L^p(0, \infty)$  and  $L^{p'}(0, \infty)$ , where

$$(4) \quad 1 < p \leq 2 < p', \quad 1/p + 1/p' = 1,$$

then all solutions of

$$(5) \quad \frac{d^2 u}{dt^2} + (a(t) + \phi(t))u = 0$$

belong to  $L^p(0, \infty)$  and  $L^{p'}(0, \infty)$ , provided that

$$(6) \quad |\phi(t)| < c_1, \quad t > t_0.$$

The most interesting case of the above theorem is  $p = p' = 2$ . For the case  $p = 1$ , the result becomes

Theorem 13. If all solutions of

$$(7) \quad \frac{d^2 u}{dt^2} + a(t)u = 0$$

are bounded and belong to  $L(0, \infty)$ , then all solutions of

$$(8) \quad \frac{d^2 u}{dt^2} + (a(t) + \phi(t))u = 0$$

belong to  $L(0, \infty)$  and are bounded, provided that

$$(9) \quad |\phi(t)| \leq c_1, \quad t \geq t_0.$$

A consequence of these theorems is the following

$$* \quad L(0, \infty) = L^1(0, \infty).$$

Theorem 14. If

$$(10) \quad |a(t)| < c_1, \quad t > t_0.$$

all solutions of (7) cannot belong to  $L^2(0, \infty)$ ; nor can they all be bounded and belong to  $L(0, \infty)$ .

For alternate proofs of the above, see Wintner, [56].

Theorem 14 was extended by Wintner, [56], to

Theorem 15. If

$$(11) \quad |a(t)| < c_1, \quad t > t_0$$

then if a solution  $u$  of (7) belongs to  $L^2(0, \infty)$ ,  $du/dt$  also belongs to  $L^2(0, \infty)$ . This is true if only

$$(12) \quad a(t) < C_1, \quad t > t_0.$$

If

$$(13) \quad (a) \quad a(t) > 0, \quad t > t_0$$

$$(b) \quad |a(t_1) - a(t_2)| < c_2 |t_1 - t_2|$$

uniformly in  $t_1$  and  $t_2$ , then all solutions of (7) cannot belong to  $L^2(0, \infty)$ .

If (11) holds, and a solution  $u$  of (7) belongs to  $L^2(0, \infty)$ ,

then  $u(t) \rightarrow 0$  as  $t \rightarrow +\infty$ ,  $u'(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , and no other linearly independent solution of (7) can remain bounded.

§. Asymptotic Behavior of Solutions of  $u'' \pm (a^2 + \phi(t))u = 0$ .

Although the asymptotic behavior of solutions of the linear system

$$(1) \quad \frac{dy}{dt} = Ay$$

has already been discussed in Chapter I, we shall consider the equation

$$(2) \quad \frac{d^2u}{dt^2} \pm (a^2 + \phi(t))u = 0$$

separately, since more precise results may be obtained. The first results in this direction are due to Poincaré, [46], and Kneser, [32]. The following result, due to Hartman, [26], is a considerable improvement of Kneser's result:

Theorem 16. Consider the equation

$$(3) \quad \frac{d^2u}{dt^2} - (1 + \phi(t))u = 0,$$

where for some  $p$  in the interval  $1 \ll p \ll 2$ ,

$$(4) \quad \int_0^{\infty} |\phi(t)|^p dt < \infty.$$

Every solution of (3) is a linear combination of two solutions,  $u_1, u_2$ , possessing the asymptotic forms,

$$(5) \quad u_1 \sim \exp \left( t + \frac{1}{2} \int_0^t \phi(s) ds \right), \quad u_1' \sim u_1,$$

$$u_2 \sim \exp \left( -t - \frac{1}{2} \int_0^t \phi(s) ds \right), \quad -u_2' \sim u_2,$$

as  $t \rightarrow +\infty$ .

A companion result is due to Wintner, [57]:

Theorem 17. If

(6) (a)  $\phi(t)$  is of bounded variation in  $(t_0, \infty)$ , and

$$(b) \quad \int_{t_0}^{\infty} |\phi(t)|^2 dt < \infty,$$

then every solution of

$$(7) \quad \frac{d^2 u}{dt^2} + (1 + \phi(t))u = 0$$

has the form

$$(8) \quad u = C_1 \cos \left( C_2 + t + \frac{1}{2} \int_0^t \phi(s) ds \right) + \varepsilon(t),$$

where  $\varepsilon(t) \rightarrow 0$ ,  $\varepsilon'(t) \rightarrow 0$ , as  $t \rightarrow \infty$ .

If  $\phi(t)$  is merely of bounded variation, then every solution of (7) has the form

$$(9) \quad u(t) = C_1 \cos \left( C_2 + \int_0^t (1 + \phi(s))^{\frac{1}{2}} ds \right) + \varepsilon(t),$$

$\varepsilon(t) \rightarrow 0$  as  $t \rightarrow +\infty$ .

If  $\phi(t)$  possesses an asymptotic series expansion as  $t \rightarrow +\infty$ ,

$$(10) \quad \phi(t) \sim \frac{a_2}{t^2} + \dots + \frac{a_n}{t^n} + \dots,$$

then the solutions will also have asymptotic representations:

Theorem 18. Every solution of

$$(11) \quad \frac{d^2 u}{dt^2} - (a^2 + \phi(t))u = 0,$$

where  $\phi(t)$  satisfies (10), is a linear combination of two solutions  $u_1, u_2$ , such that

$$(12) \quad u_1 \sim e^{at} \left( b_0 + \frac{b_1}{t} + \dots + \frac{b_n}{t^n} + \dots \right),$$

$$u_2 \sim e^{-at} \left( c_0 + \frac{c_1}{t} + \dots + \frac{c_n}{t^n} + \dots \right).$$

Every solution of

$$(13) \quad \frac{d^2 u}{dt^2} + (a^2 + \phi(t))u = 0$$

is a linear combination of two solutions,  $u_1$ ,  $u_2$ , such that

$$(14) \quad \begin{aligned} u_1 &\sim \cos at \left( d_0 + \frac{d_1}{t} + \dots + \frac{d_n}{t^n} + \dots \right) \\ u_2 &\sim \sin at \left( e_0 + \frac{e_1}{t} + \dots + \frac{e_n}{t^n} + \dots \right) \end{aligned}$$

This theorem is due to Kneser, [32], [33].

#### §7. The Liouville-Jeffries-Wentzel-Kramers-Brillouin Approximation.

The system of differential equations

$$(1) \quad \frac{dv}{dt} = -Z(t)I, \quad \frac{dI}{dt} = -Y(t)V,$$

of fundamental importance in wave-guide theory, upon eliminating  $I$ , yields the equation

$$(2) \quad \frac{d}{dt} \left( \frac{1}{Z(t)} \frac{dV}{dt} \right) = -Y(t)V$$

In this form, the equation is of the type discussed previously. However, often the equation is non-integrable; which is to say, its solution cannot be represented in terms of the classical functions of mathematical physics, associated with the names of Bessel, Hermite, Laguerre, Legendre, etc. Nevertheless, since the behavior of  $V$  as  $t \rightarrow +\infty$  is often of considerable moment, some approximate means must be devised to determine this behavior. Such a method has been obtained. In this country, it goes under the

name of the WKB (Wentzel-Kramers-Brillouin) method; in England, it is commonly called Jeffries' method, and it turns out, like so much else in the theory of differential equations, to be due to Liouville, [37]. In discussing the method, we shall follow the presentation of Schelkunoff, [49], and rigorize the procedure using a result due to Hartman.

The first step is to transform (2) into an equation of simpler type. Let

$$(3) \quad K = Z/Y, \quad L = ZY, \quad \theta = \int^t \sqrt{ZY} \quad ds,$$

where we assume that  $Z > 0$ ,  $Y > 0$ , for  $t > t_0$ . Substituting, we obtain the following equations for  $I$  and  $V$ :

$$(4) \quad \frac{d^2 V(\theta)}{d\theta^2} - \frac{K'(\theta)}{K(\theta)} \frac{dV(\theta)}{d\theta} - V(\theta) = 0$$

$$\frac{d^2 I(\theta)}{d\theta^2} + \frac{K'(\theta)}{K(\theta)} \frac{dI(\theta)}{d\theta} - I(\theta) = 0,$$

where  $K(\theta)$  now denotes the function

$$(5) \quad \frac{Z(t(\theta))}{Y(t(\theta))}$$

To eliminate the coefficients of  $dV/d\theta$  and  $dI/d\theta$ , set

$$(6) \quad V = K(\theta)^{\frac{1}{2}} \bar{V}, \quad I = K(\theta)^{-\frac{1}{2}} \bar{I},$$

thus obtaining for  $\bar{V}$  and  $\bar{I}$  :

$$(7) \quad (a) \quad \bar{V}'' - \left(1 + \frac{3K'^2}{4K^2} - \frac{K''}{2K}\right) \bar{V} = 0$$

$$(b) \quad \bar{I}'' - \left(1 - \frac{K'^2}{4K^2} + \frac{K''}{2K}\right) \bar{I} = 0.$$

At this point, we deviate from Schelkunoff's discussion and apply Hartman's result, Theorem 16. Thus

Theorem 19. If

$$(8) \quad \int_0^{\infty} \frac{(K')^{2p}}{K^{2p}} d\theta < \infty, \text{ and } \int_0^{\infty} \left| \frac{K''}{K} \right|^p d\theta < \infty,$$

for some p in the range  $1 < p < 2$ , every solution of 7(a) is a linear combination of the solutions

$$(9) \quad u_1 \sim e^{\theta + \frac{1}{2} \int_0^{\theta} \left( \frac{3(K')^2}{4K^2} - \frac{K''}{2K} \right) d\theta}$$

$$u_2 \sim e^{-\theta - \frac{1}{2} \int_0^{\theta} \left( \frac{3(K')^2}{4K^2} - \frac{K''}{2K} \right) d\theta}$$

A similar result holds for the solutions of 7(b).

§8. The Asymptotic Behavior of Solutions of  $(t^a u')' + b t^c u^n = 0.$

In the previous section, we have given a method of obtaining the asymptotic behavior of a large class of linear differential

equations. Studying the question of asymptotic behavior more closely, we consider the equation

$$(1) \quad \frac{d^2u}{dt^2} + p(t)\frac{du}{dt} + q(t)u = 0,$$

where  $p(t)$  and  $q(t)$  are asymptotic to elementary functions of the form  $t^a e^{bt}$  as  $t \rightarrow +\infty$ . Seeking to determine the behavior of the solutions with greater exactness, we may ask the following questions:

- (2)      (a) When do only monotone solutions exist?  
             (b) When do only non-monotone solutions exist?  
             (c) When do both types occur?  
             (d) How do monotone solutions behave as  $t \rightarrow +\infty$ ?  
             (e) Do non-monotone solutions behave like sine-curves?

The systematic answering of these questions was undertaken by Fowler, [23], for the equations

$$(3) \quad \frac{d}{dt} (t^a \frac{du}{dt}) + bt^c u^n = 0,$$

$$\frac{d}{dt} (e^{at} \frac{du}{dt}) + be^{at} t^c u^n = 0,$$

and for the equation

$$(4) \quad \frac{d}{dt} \left( g(t) \frac{du}{dt} \right) + f(t)u^n = 0 ,$$

where  $g(t)$  and  $f(t)$  are asymptotic to elementary functions.

Fowler's results are too extensive and intensive to be presented here, containing as they do, almost complete answers to the questions of (2).

A particular equation of the type (3) above, is Emden's equation, which is of considerable importance in astrophysics,

$$(5) \quad \frac{d}{dt} \left( t^2 \frac{du}{dt} \right) + t^3 u^n = 0 .$$

This equation, and the general class (3), were further investigated by Fowler, [21], [22], and by Sansone, [48].

#### § 9. Asymptotic Behavior - Continued.

Considering the equation

$$(1) \quad \frac{d^2u}{dt^2} + f(t)u = 0 ,$$

we may ask for simple conditions to set upon  $f(t)$  which will determine the behavior of  $u$  as  $t \rightarrow +\infty$ . One such condition is that

$$(2) \quad \int_t^{\infty} |f(t)| dt < \infty .$$

It may be shown that this implies that  $\lim_{t \rightarrow +\infty} du/dt$  exists. Actually,

this is a special case of a result due to Haupt, [27]:

Theorem 20. Consider the equation

$$(3) \quad \frac{d^n u}{dt^n} + \sum_{k=0}^{n-1} a_{n-k}(t) \frac{d^k u}{dt^k} = 0,$$

where

$$(4) \quad \int_0^{\infty} |a_k(t)| t^{k-1} dt < \infty, \quad k = 1, \dots, n$$

Then the following limit exists:

$$(5) \quad \lim_{t \rightarrow +\infty} \frac{d^{n-1} u}{dt^{n-1}}$$

A less general result was given by Caligo, [16], for  $n = 2$ . The case  $n = 2$  was proved in a simple manner by M. Boas, R. P. Boas, and Levinson, [9], and a proof of the general case, based on the same method, given by Wilkins, [52]. Another proof was given by Bellman, [5].

#### §10. Periodic Coefficients.

Consider the equation

$$(1) \quad \frac{d^2 u}{dt^2} + \phi(t)u = 0,$$

where  $\phi(t)$  is a periodic function of  $t$ . The noted equations of

Hill and Mathieu are included in this category. The general theory of this equation is not simple, and we refer to the monograph of Strutt, [50], for particulars. We are interested in conditions ensuring the boundedness of solutions of (1). From the representation theorem we know that solutions of (1) have the form

$$(2) \quad u = C_1 e^{n_1 t} P_1(t) + C_2 e^{n_2 t} P_2(t),$$

where  $n_1, n_2$  are conjugate, and  $P_1(t), P_2(t)$ , periodic functions. In what follows, we shall be interested in boundedness over the interval  $-\infty < t < \infty$ . Thus if  $u$  is to be bounded,  $n_1, n_2$  must be pure complex.

The first result in this direction was obtained by Liapounoff, [36].

Theorem 21. If

(3) (a)  $\phi(t)$  is continuous with period T,

(b)  $\phi(t) > 0,$

(c)  $\int_0^T \phi(t) dt < 4/T,$

then all solutions of (1) are bounded for  $-\infty < t < \infty$

For a discussion of the geometric significance of this result, together with a generalization, see Adamoff, [1], [2].

If the equation is written in the form

$$(4) \quad \frac{d^2 u}{dt^2} + (a + b \psi(t))u = 0,$$

where

$$(5) \quad \begin{aligned} (a) \quad & a > 0 \\ (b) \quad & \int_0^T \psi(t) dt = 0, \end{aligned}$$

it might be expected that for  $b$  small enough, the solutions would be bounded. This theory was developed by Poincaré, [3]; see Strutt, [5].

However, the general problem of the boundedness of solutions of (4) has been solved, in many important cases, by Borg, [10], using variational methods. Borg considers the following problem:

" Given equation (4), where

$$(6) \quad \begin{aligned} (a) \quad & \psi(t) \text{ has period } \pi, \\ (b) \quad & \int_0^\pi \psi(t) dt = 0 \\ (c) \quad & \left( \frac{1}{\pi} \int_0^\pi |\psi(t)|^p dt \right)^{\frac{1}{p}} = 1, \text{ for some } p \geq 1, \end{aligned}$$

to determine the regions of the (a, b) plane where all solutions of (4) are bounded in the interval  $-\infty < t < \infty$ .

Borg considers only the most important cases,  $p = 1, 2, \infty$ . The case  $p = \infty$  corresponds, by virtue of the equation

$$(7) \quad \max_{0 \leq t \leq T} |\phi(t)| = \lim_{p \rightarrow \infty} \left( \frac{1}{T} \int_0^T |\phi(t)|^p dt \right)^{\frac{1}{p}},$$

valid for continuous  $\phi(t)$ , to the condition

$$(8) \quad \max_{0 \leq t \leq T} |\psi(t)| \leq 1.$$

For  $p = 1$ , Borg derives Liapounoff's result. For  $p = 2$ , he obtains

Theorem 22. If

(9) (a)  $\phi(t)$  is continuous with period T,

(b)  $\phi(t) > 0,$

$$(c) \int_0^T \phi^2(t) dt < \frac{64}{3T^3} \left( \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2 \theta}} \right)^4,$$

the solutions of

$$(10) \quad \frac{d^2 u}{dt^2} + \phi(t)u = 0$$

are bounded in the interval  $-\infty < t < +\infty.$

The case  $p = \infty$  yields

Theorem 23. If

$$(11) \quad (a) \quad \phi(t) \text{ is continuous with period } \tau,$$

$$(b) \quad \phi(t) > 0$$

$$(c) \quad \max_{0 \leq t < \tau} |\phi(t)| < 1,$$

all solutions of (10) are bounded in the interval  $-\infty < t < +\infty$ .

Furthermore, if

$$(12) \quad 0 < a^2 \leq \phi(t) \leq b^2,$$

the necessary and sufficient condition that all solutions of (10) be bounded is that  $(a^2, b^2)$  contain no square of an integer.

As consequences of these results, Borg derives the curves defining the boundedness regions of the  $(a, b)$  plane.

For other results concerning the solutions of (1), we refer to Hamel, [25], Haupt, [27], Wiman, [54].

#### §11. Almost-periodic Coefficients.

As mentioned before, there is very little known about the solutions of equations with almost-periodic coefficients. The chief handicap is the lack of a representation theorem analogous

to that for equations with periodic coefficients. However, there is the following result due to Favard, [19]:

Theorem 24. Consider the equation

$$(1) \quad \frac{d^2u}{dt^2} + \phi(t)u = 0$$

where

- (2) (a) all solutions of (1) are bounded for  $-\infty < t < +\infty$ .  
 (b)  $\phi(t)$  is almost-periodic in the sense of Bohr,\*

Then there exists a form

$$(3) \quad F = au^2 + 2buv + cv^2,$$

a, b, c constants, which is almost-periodic, and such that  
 $F \gg k > 0$ , for all t, where u, v are two solutions of (1).

The general solution of (1) has the form

$$(4) \quad u = C_1 \sqrt{F} \cos\left(\int_{t_0}^t \frac{C}{\sqrt{F}} dt\right) + C_2 \sqrt{F} \sin\left(\int_{t_0}^t \frac{C}{\sqrt{F}} dt\right),$$

\* Which is equivalent to saying that  $\phi(t)$  is a uniform limit of trigonometric polynomials of the form  $\sum_k C_k e^{i\lambda_k t}$ , where the  $\lambda_k$  are not necessarily commensurable.

where  $C$  is a fixed constant, and  $C_1, C_2$  are arbitrary constants.

For related results pertaining to equations with quasi-periodic coefficients, see Murray, [40].

§12. Oscillation of Solutions of  $u'' + \phi(t)u = 0$ .

In addition to the question of boundedness of the solutions of

$$(1) \quad \frac{d^2u}{dt^2} + \phi(t)u = 0,$$

there is the question as to how often any particular solution vanishes in the interval  $[t_0, t]$ , as  $t \rightarrow +\infty$ . If all solutions are monotone, it is clear that each solution vanishes at most once. For further use, we introduce the following:

Definition. If a solution of (1) has an infinity of zeroes as  $t \rightarrow +\infty$ , it is said to be oscillatory; if not, it is said to be non-oscillatory.

There is a close connection between the boundedness and oscillation of solutions. However, oscillatory solutions may be unbounded, and vice versa.

The first results concerning the oscillation of solutions of second order linear differential equations were obtained by Sturm. Since then a vast body of research has arisen connected with this topic. However, since most of this has been done in connection with eigen value theory, we shall refer the interested reader to Bôcher, [8], or Ince, [30], and go on to discuss some problems connected

with the behavior of solutions of (1) over an infinite interval.

We shall begin with general properties and then, by imposing more restrictions upon  $\phi(t)$ , obtain more precise results.

Kneser, [34]\* pointed out that the following results were immediate consequences of Sturm's results:

Theorem 25. If

- (2) (a)  $\phi(t)$  is continuous for all finite  $t > t_0$ ,  
 (b)  $\lim_{t \rightarrow +\infty} \phi(t) > 0$ ,

then every solution of (1) which has a continuous first derivative is oscillatory.

Theorem 26. If

- (3) (a)  $\phi(t)$  is continuous for all finite  $t > t_0$ ,  
 (b)  $\phi(t) < 0$ ,  $t > t_0$ ,

then (1) has no oscillatory solutions. Every solution with a continuous first derivative must be monotonic and  $\rightarrow 0$  or  $\pm \infty$  as  $t \rightarrow +\infty$ .

An immediate application of this result is to the solutions of Bessel's equation

$$(4) \quad \frac{d^2 u}{dt^2} + \frac{1}{t} \frac{du}{dt} + \left(1 - \frac{n^2}{t^2}\right)u = 0.$$

\*Kneser's original statement is incorrect, cf. Fowler, [23], p. 290.

The substitution  $v = u \sqrt{t}$  yields

$$(5) \quad \frac{d^2v}{dt^2} + \left(1 + \frac{1-4p^2}{t^2}\right) v = 0,$$

which is amenable to Theorem 25.

The above theorem may be obtained by comparison with

$$(6) \quad \frac{d^2u}{dt^2} \pm a^2u = 0.$$

With this fact in mind, it is clear that if we possess sufficient information concerning the solutions of

$$(7) \quad \frac{d^2u}{dt^2} \pm a(t)u = 0$$

where  $a(t)$  is an elementary function, we may obtain other oscillation theorems. A simple choice for  $a(t)$  is  $1/4t^2$ , since the substitution  $t = e^u$  reduces (7) to an equation with constant coefficients. Thus we have the following result of Kneser, [34]:

Theorem 27. If

$$(8) \quad \begin{aligned} (a) \quad & \phi(t) \text{ is continuous for all finite } t > t_0, \\ (b) \quad & \lim_{t \rightarrow +\infty} t^2 \phi(t) < 1/4, \end{aligned}$$

all solutions of (1) will be oscillatory.

If

$$(9) \quad t^2 \phi(t) < 1/4, \quad t \geq t_0.$$

no solution will be oscillatory.

For a generalization of this result, see Hille, [29].

Kneser, [34], also considers the more general equation

$$(10) \quad \frac{d^n u}{dt^n} + \phi(t)u = 0, \quad n \geq 2.$$

Powler, [23], extended Kneser's results, and there are also extensions by Pite, [20], who considers the equation

$$(11) \quad \frac{d^n u}{dt^n} + p(t) \frac{d^{n-1} u}{dt^{n-1}} + q(t)u = 0.$$

Kneser did not use the Sturmian comparison theorems to obtain theorems 25, 26, 27, and thus was able to treat more general cases where the comparison method would fail.

§13. Oscillation of Solutions of  $u'' + f(u,t) = 0$ .

Kneser, [34], extended his results of the previous section and proved theorem 28. Consider the equation

$$(1) \quad \frac{d^2 u}{dt^2} = f(u,t),$$

where

- (2) (a)  $f(u,t)$  is continuous for all  $u$ , for  $t \geq t_0$ ,
- (b)  $f(u,t)$  has the same sign as  $u$ ,
- (c) A solution of (1) is determined by the values of  $u$  and

$u'$  at any point in the interval  $t_0 \leq t < \infty$

Then only one of the functions  $u$ ,  $u'$  can vanish, at most once, for  $t \geq t_0$ ,

Furthermore, as  $t \rightarrow \infty$ , two cases are possible,

- (3) (a)  $u \rightarrow \pm \infty$  monotonically, or  
 (b)  $u$  and  $u' \rightarrow 0$ , both monotonically, one increasing, the other decreasing.

In the simple case  $f(u, t) = f(t)u$ , we have the following result: **Theorem 29.** If

- (4) (a)  $u'' = f(t)u$   
 (b)  $f(t) > 0$ ,  $t \geq t_0$ ,  
 (c)  $f(t)$  has a continuous first derivative for  $t \geq t_0$ ,  
there exists one solution  $\rightarrow 0$  as  $t \rightarrow \infty$ , and all other such are constant multiples of the first.

This may be compared with the following result of Bôcher, [8], cf. Osgood, [41]:

**Theorem 30.** If

- (5) (a)  $u'' = f(t)u$   
 (b)  $0 < c_1 \leq f(t) \leq c_2$ ,

There is one and only one solution which remains finite, in the sense that all such are constant multiples of one, and this vanishes for  $t = +\infty$ .

This, in turn, is analogous to a theorem of Wintner's, [57]:

Theorem 31. Consider the equation

$$(6) \quad \frac{d^2u}{dt^2} + (1 + d(t))u = 0,$$

where  $d(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

If (6) has one solution  $u \rightarrow 0$  as  $t \rightarrow \infty$ , there is another solution which  $\rightarrow \infty$  as  $t \rightarrow \infty$ .

As Wintner points out, this is a consequence of the fact that the Wronskian

$$(7) \quad W(t) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

of any two solutions  $u, v$  of an equation of the form

$$(8) \quad \frac{d^2u}{dt^2} + f(t)u = 0$$

is a constant.

Using Kneser's methods, Butlewski, [11], [13], [14], extended Kneser's results, and proved

Theorem 32. Consider the equation

$$(9) \quad \frac{d}{dt} \left( e(t) \frac{du}{dt} \right) + f(u, t) = 0,$$

where

- (10) (a)  $e(t)$  is differentiable for  $t \geq t_0$ ,  
 (b)  $e(t) > 0$ ,  $t \geq t_0$ ,  $\lim_{t \rightarrow \infty} 1/e(t) > 0$ .  
 (c)  $f(u, t)$  is continuous in  $u$  and  $t$  for  $t \geq t_0$ ,

- (d) A solution  $u$  of (9) is determined by the values of  $u$  and  $u'$  at any point in the interval  $\infty > t \geq t_0$ .
- (e)  $f(u,t)$  has the sign of  $u$ .
- (f) If  $\lim_{t \rightarrow \infty} u(t) = g > 0$ ,  $\lim_{t \rightarrow \infty} f(u,t) > 0$ ,
- If  $\lim_{t \rightarrow \infty} u(t) = g < 0$ ,  $\lim_{t \rightarrow \infty} f(u,t) < 0$ ,

Then every solution of (9) is oscillating.

The following result was obtained by Picard, [45]:

Theorem 33. If

- (11) (a)  $\theta(t) \equiv 1$ ,
- (b)  $f(u,t)$  increases constantly as  $u$  does,  $u > 0$ ,
- (c)  $f(0,t) \equiv 0$ ,
- (d)  $\frac{\partial}{\partial u} f(u,t) > 0$ , and decreases as  $u$  increases,  $u > 0$ ,
- (e)  $-f(-u,t)$  has same properties as  $f(u,t)$ ,

every solution of (9) is oscillating.

The following result was obtained by Milne, [39]:

Theorem 34. Consider the equation

$$(12) \quad \frac{d^2u}{dt^2} + d(t)f(u) = 0$$

where

- (13) (a)  $d(t)$  is positive, continuous, monotone increasing,  
and bounded for  $t \geq t_0$ ,
- (b)  $f(u)$  is odd, monotone increasing, and  $|f(u_1) - f(u_2)|$   
 $\leq c_1|u_1 - u_2|$ , for  $-a \leq u_1, u_2 \leq a$ ,  $a > 0$ ,

$$(c) \frac{du}{dt} = 0, \quad u = u_1, \quad \text{for } t = t_0 > t_0,$$

$$(d) |u_1| < a, \quad f(u_1) = 0.$$

Then  $u$  is oscillatory, and its amplitude decreases monotonically, but does not approach zero.

Butlewski, [13], pointed out that results similar to those of theorem 32 can be obtained for

$$(14) \quad \frac{d}{dt} \left( e_{n-1}(t) \frac{d}{dt} \left( e_{n-2}(t) \dots \frac{d}{dt} \left( e(t) \frac{du}{dt} \dots \right) + f(u, t) = 0. \right. \right.$$

As corollaries of theorem 32, we have the following results:

Theorem 35. Consider the equation

$$(15) \quad \frac{d}{dt} \left( e(t) \frac{du}{dt} \right) + \sum_{i=0}^n \phi_{2i+1}(t) u^{2i+1} = 0,$$

where

- (16) (a)  $e(t)$  is differentiable for  $t \geq t_0$ ,  
 (b)  $\phi_{2i+1}(t)$  is continuous and positive for  $t \geq t_0$ ,  
 (c)  $\lim_{t \rightarrow \infty} 1/e(t) > 0,$   
 (d)  $\lim_{t \rightarrow \infty} \phi_{2i+1}(t) > 0.$

Then every solution is oscillatory.

Theorem 36. Consider the equation

$$(17) \quad \frac{d}{dt} \left( e(t) \frac{du}{dt} \right) + \phi(t) f(u) = 0,$$

where

- (18) (a)  $\lim_{t \rightarrow \infty} 1/\theta(t) > 0,$   
 (b)  $\theta(t)$  is differentiable for  $t \geq t_0,$   
 (c)  $\lim_{t \rightarrow \infty} \phi(t) > 0,$   
 (d)  $\lim_{u \rightarrow +\infty} f(u) > 0, \quad \lim_{u \rightarrow -\infty} f(u) < 0,$   
 (e)  $f(u)$  is continuous for all finite  $u, \phi(t)$  for  $t \geq t_0.$

Then every solution is oscillating.

The equation

$$(19) \quad \frac{d^2x}{dt^2} + f(x) = g(\sin \omega t)$$

is important physically. For reference to previous work on this equation, and its physical origin, we refer to John [31],\* where various boundedness and oscillation properties of the solution are given.

Butlewski [14] also obtains results for non-oscillation, which generalize theorem 28. He also investigated the zeroes of solutions of systems of differential equations of the type

$$(20) \quad \begin{aligned} \frac{dy_1}{dt} &= a_{11}(t)y_1 + a_{12}(t)y_2 \\ \frac{dy_2}{dt} &= a_{21}(t)y_1 + a_{22}(t)y_2, \end{aligned}$$

Butlewski, [14].

§14 Magnitude of Oscillations of Solutions of  $u'' + \phi(t)u = 0.$

In this section, we shall discuss the equation

\*For a comprehensive report, see Friedrichs and Stoker, [27].

$$(1) \quad \frac{d^2 u}{dt^2} + \phi(t)u = 0$$

with particular reference to the following questions:

- (2) (a) What are the magnitude and frequency of the oscillations of solutions of (1)?  
 (b) What is the magnitude of  $du/dt$ ?

By imposing some conditions upon  $\phi(t)$  which are satisfied for a large class of elementary functions, these questions may be answered completely. However, we shall begin with some general results first.

Theorem 37. If

- (3) (a)  $\phi(t) > 0$ ,  $t \geq t_0$ ,  $\phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ,  
 (b)  $\phi(t)$  is monotone increasing,

then every solution of (1)  $\rightarrow 0$  as  $t \rightarrow \infty$ .

This result is due to Armellini, [3].

If an additional condition is placed upon  $\phi$ , we can estimate the order of smallness of  $u$  as  $t \rightarrow \infty$ . Thus:

Theorem 38. If

- (4) (a)  $\phi'(t) > 0$ ,  $t > t_0$ ,  
 (b)  $\phi'(t)$  is non-increasing,  
 (c)  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ ,

every solution of (1)  $\rightarrow 0$  as  $t \rightarrow \infty$ . However,  $\overline{\lim}_{t \rightarrow \infty} |u(t)\sqrt{\phi(t)}|$

is positive.

By analogy with the equation

$$(5) \quad \frac{d^2u}{dt^2} \pm a^2u = 0,$$

we might expect that the quantity  $\sqrt{\phi(t)}$  would play an important role in determining the oscillation of the solution. This is actually so. Let us begin the discussion with the following result due to Petrovitch, [44]:

Theorem 39. Consider the equation

$$(6) \quad \frac{d^2u}{dt^2} = \phi(t)u,$$

in which  $\phi(t) > 0$  for  $a \leq t \leq b$ . Let  $u$  be the solution satisfying the boundary condition  $u'(t_0) = 0$ ,  $a \leq t_0 \leq b$ . Then  $u$  may be written

$$(7) \quad u = \frac{e^T + e^{-T}}{2},$$

where

$$(8) \quad T = (t - t_0) \sqrt{\phi(s)}, \quad t_0 \leq s \leq t,$$

and  $s$  depends on  $t$

Thus the solution is comprised between

$$(9) \quad \frac{e^{T_1} + e^{-T_1}}{2} \quad \text{and} \quad \frac{e^{T_2} + e^{-T_2}}{2},$$

where

$$(10) \quad T_1 = (t - t_0) \sqrt{\lim \phi(s)}, \quad T_2 = (t - t_0) \sqrt{\lim \phi(s)},$$

$$t_0 \leq s \leq t.$$

If  $\phi(t) < 0$ , the solution in the interval between two successive zeroes,  $t_1, t_2$ , has the form

$$(11) \quad u = \cos T, \quad T = (t - t_0) \sqrt{-\phi(s)}, \quad t_1 \leq s \leq t_2$$

Thus

$$(12) \quad t_1 = t_0 - \frac{\pi}{2\sqrt{-\phi(s)}}, \quad t_2 = t_0 + \frac{\pi}{2\sqrt{-\phi(s)}}$$

Hence the length  $l_1$  of a half-wave satisfies the inequalities

$$(13) \quad \frac{\pi}{\sqrt{\lim(-\phi(s))}} \leq l_1 \leq \frac{\pi}{\sqrt{\lim(-\phi(s))}}, \quad t_1 \leq s \leq t_2.$$

In connection with the above results, the following results are interesting. The first is due to Osgood, [41], the second to Murray, [46].

Theorem 40. If

$$(14) \quad \phi(t) \geq 0, \quad t \geq t_0,$$

the general solution of (6) has the form

$$(15) \quad u = c_1 e^{\int_{t_0}^t \lambda_1(t_1) dt_1} + c_2 e^{\int_{t_0}^t \lambda_2(t_1) dt_1}$$

where  $\lambda_1(t), \lambda_2(t) \geq 0$ ,  $t \geq t_0$ , and bounded if  $\phi(t)$  is.

Theorem 41. If

- (16) (a)  $\phi(t) > 0$ ,  
 (b)  $\phi(t)$  is monotone,

the amplitudes of solutions of (1) vary monotonically, increasing when  $\phi(t)$  is a decreasing function, decreasing otherwise. Furthermore if  $\phi(t)$  remains finite as  $t \rightarrow \infty$ , the amplitudes remain above a certain bound, depending upon  $u(0)$ .

Ascoli, [4], considerably extended Osgood's results and proved the following:

Theorem 42. If

- (17) (a)  $\phi(t)$  is monotone,  
 (b)  $\phi(t) \rightarrow a^2$  as  $t \rightarrow \infty$ ,

then, if  $u$  satisfies (1)

$$(18) \quad \lim_{t \rightarrow \infty} \max_{0 \leq s \leq t} |u| = c_1$$

$$\lim_{t \rightarrow \infty} \max_{0 \leq s \leq t} |u'| = c_2,$$

and  $c_2 = ac_1$ .

If  $\phi(t)$  is non-decreasing,  $\max_{0 \leq s \leq t} |u'|$  is non-increasing, and approaches a finite limit as  $t \rightarrow \infty$ ,  $\max_{0 \leq s \leq t} |u'|$  is non-decreasing, but may approach  $\infty$  as  $t \rightarrow \infty$ .

If  $\phi(t)$  is non-increasing, the above results hold with  $u$  and  $u'$  interchanged.

Theorem 39 shows the importance of the quantity  $\sqrt{\pm \phi(t)}$ , and we shall now discuss more precise results. Biernacki, [7], showed

Theorem 43. If

- (19) (a)  $\phi'(t) > 0, t \geq t_0,$   
 (b)  $\phi'(t)$  is non-decreasing,  
 (c)  $\phi(t + 1/\sqrt{\phi(t)}) / \phi(t) \rightarrow 1$  as  $t \rightarrow \infty,$

every solution of (1)  $\rightarrow 0$  as  $t \rightarrow \infty$ . However,  $\lim_{t \rightarrow \infty} |u(t) \sqrt{\phi(t)}|$  is positive.

Milloux, [8], took up this question using the transformation

$$(20) \quad u = r \cos \theta, \quad \theta = c \int \frac{dt}{r^2},$$

used by Fatou, [9], which reduces (1) to the non-linear equation

$$(21) \quad \frac{d^2 r}{dt^2} - \frac{c^2}{r^3} + r \phi(t) = 0$$

His result is

Theorem 44. If  $u$  satisfies (1), and

$$(22) \quad \phi(t) \rightarrow \infty \text{ as } t \rightarrow \infty,$$

then for intervals  $(t_1, t_2)$ , such that

$$(23) \quad \phi(t_2) / \phi(t_1) \rightarrow 1 \text{ as } t_2, t_1 \rightarrow \infty$$

we have

$$(24) \quad u = \sqrt{c} \phi(t)^{-1/4} (1 + \varepsilon(t)), \quad t_1 \leq t \leq t_2,$$

where  $\varepsilon(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Hence  $\overline{\lim}_{t \rightarrow \infty} |\phi(t)^{1/4} u| > \sqrt{c}$ .

Continuing in this vein, Wiman, [54], imposed the following condition upon  $\phi(t)$ :

Condition A.

$$(25) \quad \lim_{t \rightarrow \infty} \frac{\phi\left(t + \frac{c}{\phi(t)}\right)}{\phi(t)} = 1$$

This holds if, for example,  $\phi' / \phi^{3/2} \rightarrow 0$  as  $t \rightarrow \infty$ .

Using this condition Wiman, [54], proved

Theorem 45. If condition A is satisfied, where  $\phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , then if  $\Delta(t)$  is the interval between two successive zeroes of a solution of (1), we have

$$(26) \quad \lim_{t \rightarrow \infty} \frac{\Delta(t) \sqrt{\phi(t)}}{\pi} = 1$$

Furthermore, if both  $\phi$  and  $\phi'$  satisfy condition A, the amplitude of  $u$  is of order  $\phi(t)^{-1/4}$ , and that of  $u'$  of order  $\phi(t)^{1/4}$  at the zeroes of  $u$ , as  $t \rightarrow \infty$ .

Related weaker results were proved by Horn,

Using Condition A, Wiman proved:

Theorem 46. If condition A is satisfied, and  $u$  satisfies

$$(27) \quad u'' = \phi(t)u,$$

where  $\phi(t) > 0$ , then

$$(28) \quad \lim_{t \rightarrow \infty} \frac{u'}{u \sqrt{\phi(t)}} = 1,$$

in general, but there exist solutions for which

$$(29) \quad \lim_{t \rightarrow \infty} \frac{u'}{u \sqrt{\phi(t)}} = -1$$

For further results on the magnitude of the oscillations, we refer to Biernacki, [7], who considers non-linear equations, Fowler, [23], Milloux, [33], Wiman, [54], [ ], and Wintner, [55].

#### §15 Non-Oscillation Theorems.

We now consider some conditions which ensure that no solution of

$$(1) \quad \frac{d^2 u}{dt^2} + \phi(t)u = 0$$

are oscillatory.

One of the principal tools in the study of the linear equation (1) is the non-linear Riccati equation

$$(2) \quad \frac{dv}{dt} + v^2 + \phi(t) = 0,$$

satisfied by  $v = u'/u$ .

The connection between the two has been known ever since Euler, and (2) was used by Poincaré, [46]. In the form of the non-

linear integral equation

$$(3) \quad w = t \int_t^{\infty} \frac{w^2}{t^2} dt + t \int_t^{\infty} \phi(t) dt, \quad w = tv,$$

it is used by Hille, [29], to derive the results below.

Theorem 47. If (1) admits one solution such that

$$(4) \quad \lim_{t \rightarrow \infty} u = 1,$$

then its general solution has the form

$$(5) \quad u = c_1(1 + \varepsilon_1(t)) + c_2 t(1 + \varepsilon_2(t)),$$

where  $\varepsilon_1, \varepsilon_2 \rightarrow 0$  as  $t \rightarrow \infty$ .

Thus, every solution is non-oscillatory.

Theorem 48. If

$$(6) \quad \int_t^{\infty} t|\phi(t)| dt < \infty,$$

there is a solution u of (1) such that  $\lim_{t \rightarrow \infty} u = 1$ . Moreover,

$$(7) \quad |u - 1| \leq \left( \exp \int_t^{\infty} t|\phi| dt - 1 \right).$$

Conversely, if  $\phi(t)$  has constant sign, and there exists a solution such that  $\lim_{t \rightarrow \infty} u \rightarrow 1$ , then  $\phi(t)$  satisfies (6).

Theorem 49. If

$$(8) \quad \int_0^{\infty} t^2 |\phi(t)| dt < \infty,$$

there is a solution u of (1) such that  $\lim_{t \rightarrow \infty} (u - t) = 0$ .

If  $\phi(t)$  has constant sign for large t, (8) is necessary and sufficient that  $\lim_{t \rightarrow \infty} (u - t) = 0$ .

Theorem 50. If

$$(9) \quad \begin{aligned} (a) \quad & \mu(t) \text{ is a positive non-decreasing function,} \\ (b) \quad & \int_0^{\infty} \frac{\mu(t)}{t^2} dt < \infty, \end{aligned}$$

then if (1) has non-oscillatory solutions, we must have

$$(10) \quad \int_0^{\infty} \mu(t) |\phi(t)| dt < \infty$$

Theorem 51. Consider the two equations

$$(11) \quad \begin{aligned} (a) \quad & u'' + \phi_1(t)u = 0, \\ (b) \quad & v'' + \phi_2(t)v = 0, \end{aligned}$$

Define

$$(12) \quad \begin{aligned} g_1(t) &= t \int_t^{\infty} \phi_1(t) dt \\ g_2(t) &= t \int_t^{\infty} \phi_2(t) dt \end{aligned}$$

If the solutions of (a) are non-oscillatory, and

$$(13) \quad g_1 \geq g_2, \quad t \geq t_0,$$

then the solutions of (b) are non-oscillatory.

15. Non-Oscillation Theorems continued. Consider the equations

$$(1) \quad (a) \quad \frac{d^2u}{dt^2} - p_1(t) \frac{du}{dt} - p_2(t)u - q(t) = 0, \quad t \geq t_0,$$

$$(b) \quad \frac{d^2v}{dt^2} - p_1(t) \frac{dv}{dt} - p_2(t)v - q(t) > 0,$$

where  $u(t_0) = u_0 = v(t_0)$ ,  $u'(t_0) = u'_0 = v'(t_0)$ , and

$p_1, p_2, q$  are continuous functions for  $t > t_0$ .

It is clear that  $v > u$  in some interval  $t_1 > t > t_0$ , and the question arises as to the length of this interval. The original result is due to Tchaplygin, and a partial converse to Petrov, [43]. The result below gives the best possible bound, and is due to Wilkins, [53].

Theorem 52. Let  $u, v$  satisfy 1(a) and 1(b). Then  $v > u$  for  $t_1 > t > t_0$ , provided there exists a solution  $u$  of 1(a), which does not vanish for  $t_1 > t > t_0$ .

A related result is the following theorem due to Polya, [47]:

Theorem 53. Let the equation

$$(2) \quad L(u) = u^{(n)} + p_1(t) u^{(n-1)} + \dots + p_n(t) u = 0,$$

possess the following property:

Property W: There exist  $(n - 1)$  solutions  $u_1, u_2, \dots, u_{n-1}$ , of  
 (2) such that

$$(3) \quad u_1 > 0, \quad W(u_1, u_2) > 0, \quad \dots, \quad W(u_1, u_2, \dots, u_{n-1}) > 0,$$

$$a < t < b,$$

where

$$(4) \quad W(u_1, u_2, \dots, u_k) = \begin{vmatrix} u_1 & u_1' & \dots & u_1^{(k-1)} \\ u_2 & u_2' & & \\ \vdots & & & \\ u_k & u_k' & & u_k^{(k-1)} \end{vmatrix}$$

Then if  $u(t)$  vanishes at  $(n+1)$  points in  $(a, b)$ , there  
exists an intermediate point such that

$$(5) \quad L(u(\xi)) = 0.$$

A somewhat related result, which is a generalization of the uniqueness theorem for differential equations is due to Pite, [20].

Theorem 54. Consider the system

$$(6) \quad \frac{dy_1}{dt} = f_1(y_1, y_2, \dots, y_n, t),$$

where

$$(7) \quad |f_1(t, y) - f_1(t, z)| \leq \sum_{k=1}^n l_k |y_k - z_k|,$$

for  $y_k, z_k$  arbitrary. Let  $(y_1, y_2, \dots, y_n), (z_1, z_2, \dots, z_n)$  be two

solutions of (1) such that

$$(8) \quad y_k(t_k) = z_k(t_k), \quad k = 1, 2, \dots, n, \quad a \leq t_k \leq b,$$

where the  $t_k$  are any  $n$  points of the interval  $(a, b)$  of length  $p$ .

Then

$$(9) \quad p > \frac{1}{\sum_{k=1}^n l_k}$$

Hence if

$$(10) \quad f_1(t, 0, 0, \dots, 0) = 0,$$

no non-identically vanishing solution of (6) can have all its components vanish individually at points inside an interval  $(a, b)$  of length  $p$ , if  $p$  satisfies (9).

Since every  $n$ -th order linear differential equation can be converted into a system of the type (6), the above result can be interpreted in terms of the vanishing of the solutions and its derivatives, cf. Fite, [2].

If the  $l_k$  are functions of  $t$ ,  $l_k(t)$ , condition (9) can be replaced by

$$(11) \quad \sum_{k=1}^n \int_a^b l_k(t) dt > 1.$$

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1. Introduction

In this chapter, we shall consider the behavior of solutions of difference equations of the type

$$(1) \quad z(t+1) = F(z(t), t), \quad t = 0, 1, 2, \dots$$

Here, as in the case of differential equations,  $z(t)$  is an  $n$ -dimensional column vector, and  $F(z(t), t)$  is a vector function of  $z(t)$ .

A problem which has been extensively investigated in the modern theory of difference equations is the question of the existence of analytic solutions of equations such as (1). There is an extensive literature on this subject, cf. Norlund, [9], Trjitzinsky, [16]. Recurrence relations of the type (1) occur very frequently in applied mathematics, and then the question of interest is usually the behavior of  $z(t)$  as  $t \rightarrow \infty$ . If the equation is non-linear, it is, in general, impossible to solve for  $z(t)$  in terms of elementary functions, and, as in the case of differential equations, recourse must be had to other means.

The technique of power series solutions of differential equations generates recurrence relations of the above form. If we consider the differential equation

$$(2) \quad \frac{dz}{dt} = f(z), \quad z(0) = z_0,$$

and try a solution of the form

$$(3) \quad z = z_0 + \sum_{n=1}^{\infty} y(n)t^n,$$

upon equating coefficients of  $t^n$ , we obtain an equation of type (1) for  $y(n)$ . This equation, however, is slightly different from the one we shall consider below, since the form of  $F(z,t)$  changes with  $t$ .

A different technique applied to the differential equation (2) yields precisely the form of equation we shall study. Let us try to approximate to the solutions of (2) by means of the solutions of

$$(4) \quad \frac{z(t+h) - z(t)}{h} = f(z), \quad t = 0, h, 2h, \dots$$

Since difference equations are essentially simpler to handle than differential equations, the equation in (4) can be used to derive many of the properties of the solutions of (2), cf. Bellman, [1].\*

We shall treat the case where  $F(z(t),t)$  is approximately linear, that is,

$$(5) \quad F(z(t),t) = A(t)z + f(z,t),$$

where

$$(6) \quad \|f(z,t)\| / \|z\| \leq c_1,$$

and  $c_1$  is a "small" constant. The theory is completely parallel to that for differential equations, and in light of the relation between (2) and (4), that is not very surprising.

\* The idea of approximating to a differential equation by a difference equation is a very old one, and has been used by many authors.

As in the case of differential equations, the equation

$$(7) \quad y(t + 1) = Ay(t),$$

where  $A$  is a constant matrix can be completely solved in terms of elementary functions. To treat the non-linear equation

$$(8) \quad z(t + 1) = Az(t) + f(z),$$

we shall express  $z$  in terms of  $y$ . Here a peculiar difficulty arises. This may only be done if  $\det A \neq 0$ , or, equivalently, if  $A$  has no zero characteristic roots. If  $\det A = 0$ , we see from the matrix equation

$$(9) \quad Y(t + 1) = AY(t), \quad Y(0) = I,$$

that  $Y(t)$  is singular for every  $t > 0$ . This is in contrast to the state of affairs for differential equations, where  $Y(t)$  is never singular, as long as  $A(t)$  is integrable. This difficulty is not serious. The zero characteristic roots may be isolated, and the final results are analogous to those for differential equations.

2. The equation  $y(t + 1) = Ay(t)$ ,  $A$  a constant matrix.

To solve the equation

$$(1) \quad y(t + 1) = Ay(t), \quad t = 0, 1, 2, \dots,$$

where  $A$  is constant, set  $y(t) = c \lambda^t$ , where  $c$  is a constant vector, and  $\lambda$  is a complex constant. Upon eliminating  $\lambda^t$ , we obtain

$$(2) \quad \lambda c = Ac,$$

whence if  $c$  is non-trivial,

$$(3) \quad |A - \lambda I| = 0,$$

the familiar characteristic equation. To every root  $\lambda$  of (3) there corresponds a solution of (1),

$$(4) \quad y = c(\lambda) \lambda^t.$$

If  $\lambda$  is a multiple root of multiplicity  $k$ , there will correspond  $k$  solutions of the type, (in general),

$$(5) \quad c_1(\lambda) \lambda^t, c_2(\lambda) t \lambda^t, \dots, c_k(\lambda) t^{k-1} \lambda^t.$$

If  $\lambda$  is complex, the real and imaginary parts of the above solutions furnish the real solutions of (1).

If  $\det A \neq 0$ , the  $n$  solutions found in this way constitute a fundamental set. Let  $Y$  be the solution of

$$(6) \quad Y(t+1) = AY(t), \quad Y(0) = I.$$

Then  $Y(t)$  is unique; for if  $W = YU$  is another solution,

$$(7) \quad Y(t+1)U(t+1) = AY(t)U(t),$$

or since  $Y(t)$  is non-singular,

$$(8) \quad U(t+1) = U(t) = U(0).$$

If  $A$  has zero characteristic roots, we may put  $A$  into the

form

$$(9) \quad A = \begin{pmatrix} L_1 & 0 \\ 0 & B \end{pmatrix},$$

where  $L_1$  is a matrix of order  $k$  corresponding to a  $k$ -multiple zero root.

Since  $L_1$  is composed of matrices of the type,

$$(10) \quad \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

along its diagonal and  $\det B \neq 0$ , the solution of (6) will have the form

$$(11) \quad Y(t) = \begin{pmatrix} 0 & 0 \\ 0 & Y_1(t) \end{pmatrix},$$

where  $Y_1$  is non-singular. Hence, if we are considering the linear case, the zero characteristic roots may be ignored. However, in treating the perturbed linear equation

$$(12) \quad z(t+1) = (A + B(t))z(t),$$

or the non-linear case

$$(13) \quad z(t+1) = Az(t) + f(z,t),$$

the zero characteristic roots must be considered.

From the form of the solution it is seen that the circumference of the unit circle in the complex plane is a boundary of two regions of importance in the discussion of the boundedness of solutions of the difference equation. To roots lying inside the unit circle there correspond manifolds of solutions tending to zero as  $t \rightarrow \infty$ ; to roots lying outside the unit circle, there correspond manifolds of solutions tending to  $\infty$  as  $t \rightarrow \infty$ ; roots on the unit circle present either case, depending upon the form of their elementary divisors.

### 3. An Important Lemma

Let us consider the two equations

$$(1) \quad y(t+1) = A y(t)$$

$$(2) \quad z(t+1) = A z(t) + w(t)$$

where  $\det A \neq 0$  and  $A$  is a constant matrix. The following result is then valid.

Lemma 1. We have

$$(3) \quad z(t+1) = y(t+1) + \sum_{t_1=0}^t Y(t-t_1) W(t_1),$$

where  $y(t)$  is the solution of (1) with the same initial value as  $z$ , and  $Y$  is the solution of

$$(4) \quad Y(t+1) = AY(t), \quad Y(0) = I$$

Proof: We use the method of variation of parameters. Let

$$(5) \quad z = Y u$$

Then

$$(6) \quad Y(t+1) u(t+1) = A Y(t) u(t) + w(t),$$

and since  $Y(t+1) = A Y(t)$ ,  $\det Y \neq 0$ ,

$$(7) \quad u(t+1) = u(t) + Y^{-1}(t+1) w(t).$$

Thus

$$(8) \quad u(t+1) = u(0) + \sum_{t_1=0}^t Y^{-1}(t_1+1) w(t_1);$$

whence

$$(9) \quad z(t+1) = Y u = y(t+1) + \sum_{t_1=0}^t Y(t+1) Y^{-1}(t_1+1) w(t_1).$$

This is the general result that holds for  $A$  a variable matrix which is non-singular. If  $A$  is constant however, we shall show that

$$(10) \quad Y(t+1) Y^{-1}(t_1+1) = Y(t-t_1).$$

This follows from the uniqueness of the solution of (4). The right side is a solution with the value  $I$  at  $t = t_1$ , and the same is true of the left side. Thus equality for all  $t$ .

If  $A$  has zero characteristic roots, we write it

$$(11) \quad A = C^{-1} \begin{pmatrix} L_1 & 0 \\ 0 & B \end{pmatrix} C$$

where  $C$  is a non-singular elementary matrix,  $\det B \neq 0$ , and  $L_1$  is the matrix corresponding to the multiple zero root. The change of variable  $Cz = \bar{z}$  transforms (2) into

$$(12) \quad \bar{z}(t+1) = \begin{pmatrix} L_1 & 0 \\ 0 & B \end{pmatrix} \bar{z}(t) + Cw(t).$$

Now decompose  $\bar{z}$  into the two vectors  $\bar{z}_1, \bar{z}_2$ , where  $\bar{z}_1$  is a  $k$ -dimensional column vector,  $k$  the multiplicity of the zero characteristic root,

$$(13) \quad \bar{z} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}$$

then (12) becomes

$$(14) \quad \bar{z}_1(t+1) = L_1 \bar{z}_1 + \bar{w}_1$$

$$(15) \quad \bar{z}_2(t+1) = B \bar{z}_2 + \bar{w}_2,$$

where

$$(16) \quad Cw = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix}$$

Equation (15) may now be treated by means of Lemma 1, and (14), because of the special form of  $L_1$ , may be handled directly.

#### 4. The Linear Equation $z(t+1) = (A+B(t))z(t)$ .

The analogue of Hukuwara's theorem for differential equations, Theorem 4 of Chapter 1, is valid.

Theorem 1. If

- (1) (a) A is a constant matrix,  
 (b)  $\sum_{t=0}^{\infty} \|B(t)\| < \infty,$   
 (c) all solutions of  $y(t+1) = Ay(t)$  are bounded,  
then all solutions of

$$(2) \quad z(t+1) = (A + B(t)) z(t),$$

are bounded.

There arise extra complications due to the possibility of  $\det A$  being zero, but these may be taken care of without undue difficulty.

5. The Non-Linear Equation  $z(t+1) = Az(t) + f(z)$  : I

Before proceeding to the discussion of the boundedness of the solutions of the non-linear equation when  $A$  is a constant, we shall exhibit an example due to Ta Li, [15], an analogue of one of Perron for differential equations, illustrating the dangers of intuition.

Theorem 2. There exists equations of the form

$$(1) \quad y(t+1) = A(t) y(t),$$

with  $\|A(t)\|$  bounded, with the property that every solution of

(1)  $\rightarrow 0$  as  $t \rightarrow \infty$ , and such that not every solution of

$$(2) \quad z(t+1) = A(t)z(t) + f(z)$$

where  $f(z)$  is a non-linear term, tends to zero as  $t \rightarrow \infty$ , and such that there exist unbounded solutions of (2).

An example is

$$(3) \quad y_1(t+1) = e^{-a} y_1(t), \quad 1/2 < a < \frac{1+\sqrt{5}}{2},$$

$$y_2(t+1) = e^{(\sin \log(t+1) - 2a)(t+1) - (\sin \log t - 2a)t} y_2(t),$$

$$z_1(t+1) = e^{-a} z_1(t)$$

$$z_2(t+1) = e^{(\sin \log(t+1) - 2a)(t+1) - (\sin \log t - 2a)t} z_2(t) + z_1^2(t).$$

6. The Non-Linear Equation  $z(t+1) = A_z(t) + f(z)$  : II..

The following result is due to Perron, [10]:

Theorem 3. If  $k, k \leq n$ , of the characteristic roots of  $A$  lie inside the unit circle, there is a  $k$ -dimensional manifold of solutions of

$$(1) \quad z(t+1) = A z(t) + f(z)$$

which  $\rightarrow 0$  as  $t \rightarrow \infty$ , provided that

$$(2) \quad \|f(z)\| / \|z\| \rightarrow 0$$

as  $\|z\| \rightarrow 0$ .

If all characteristic roots lie inside the unit circle, every solution of (1) for which  $\|z(0)\|$  is sufficiently small

approaches zero as  $t \rightarrow \infty$ , provided that (2) holds.

Similarly, we have, Bellman, [1]:

Theorem 4. If all solutions of

$$(3) \quad y(t+1) = A y(t),$$

A a constant matrix, are bounded, then all solutions of

$$(4) \quad z(t+1) = A z(t) + f(z,t)$$

are bounded, provided that

$$(5) \quad (a) \quad \|f(z,t)\| / \|z\| \leq g(t), \quad \|z\| \leq c_1,$$

$$(b) \quad \sum_{t=0}^{\infty} g(t) dt < \infty,$$

$$(c) \quad \|z(0)\| \text{ is sufficiently small.}$$

For a further discussion of non-linear difference equations, containing results corresponding to theorem 8 of Chapter 2, we refer to Ta Li, [15].

#### 7. A asymptotic Behavior of Solutions.

Since all the classical orthogonal polynomials, e.g. Legendre, Laguerre, Hermite, Jacobi, satisfy recurrence relations of the type

$$(1) \quad P_n + a_1(n, x) P_{n-1} + a_2(n, x) P_{n-2} = 0,$$

the importance of an investigation of the limit

$$(2) \quad \lim_{n \rightarrow \infty} \frac{P_n}{P_{n-1}}$$

by means of the theory of difference equations, to the theory of the convergence of the orthogonal series

$$(3) \quad \sum_{n=1}^{\infty} a_n P_n$$

is readily seen.

This investigation was begun by Poincaré, [12], and continued by Perron, [11]. Independently of Perron, Ford, [4], using methods of Dini, [3], began the study of the more general question of the asymptotic behavior of solutions of difference equations of the form

$$(4) \quad z(t+1) = (A + B(t)) z(t),$$

where  $A$  is a constant matrix, and  $\|B(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ .

We shall begin by presenting Poincaré's original result.

Theorem 4. Consider the difference equation

$$(5) \quad u(n+t) + \sum_{k=0}^{n-1} a_k(t) u(t+k) = 0,$$

where

$$(6) \quad (a) \quad \lim_{t \rightarrow \infty} a_k(t) = a_k,$$

$$(b) \quad \text{the equation } r^n + \sum_{k=0}^{n-1} a_k r^k = 0 \text{ has}$$

all its roots real and distinct.

Then the limit

$$(7) \quad \lim_{t \rightarrow \infty} \frac{u(t+1)}{u(t)} = r_1,$$

exists and  $r_1$  is a root of the equation of 6(b).

There are generalizations of this result due to Perron.  
For these results, and references, we refer to Norlund, [9].

If instead of a result such as (7), we wish a result of the form

$$(8) \quad u(t) \sim r_1^t$$

it is necessary to know something of the order of magnitude of  $a_k - a_k(t)$  as  $t \rightarrow \infty$ . The following result is due to Ford, [5]:

Theorem 5. Consider the difference equation of (5), where

$$(9) \quad (a) \quad |a_k - a_k(t)| \leq \phi(t), \quad \text{as } t \rightarrow \infty,$$

$$(b) \quad \sum_t^{\infty} \phi(t) < \infty,$$

$$(c) \quad \text{the roots of } r^n + \sum_{k=0}^{n-1} a_k r^k = 0, \text{ real or complex, are distinct.}$$

If all the roots have the same modulus,

$$(10) \quad u(t) = \sum_{i=1}^n c_i r_i^t + |r_1|^t \varepsilon(t),$$

where  $\varepsilon(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

This result may be extended to obtain a result corresponding to Theorem 31 of Chapter I.

The second order equation

$$(11) \quad u(x+2) + a_0(x) u(x+1) + a_1(x) u(x) = 0$$

was considered in detail by Ford, [5], where other references are given. As application, Ford shows that various asymptotic formulas for the Legendre polynomials may be derived.

#### 8. Magnitude of Solutions of Non-Linear Difference Equations.

Results analogous to those obtained for differential equations have been obtained by Lancaster, [7], and Shah [13].

#### 9. Difference Equations with Arbitrary Real Spans.

The difference equation

$$(1) \quad \sum_{s=1}^n a_s(t) u(t+d_s) = 0$$

where the  $d_s$  are real but not commensurable is much more difficult. Generalizations of Poincaré's and Perron's results to this case have been given by Bochner, [2], and Martin, [8].

The study of equation (1) requires much more complicated mathematical apparatus than the case treated previously. The limiting case of (1), namely, differential-difference equations of the type

$$(2) \quad \frac{d^n u}{dt^n}(t+d_n) + \sum_{j=1}^n \sum_{i=0}^{n-1} a_{ij}(t) \frac{d^i u}{dt^i}(t+d_j) = 0$$

has been studied by Hilb, [6], where further references are given.

The non-linear equations corresponding to (1) and (2) have been studied by Bellman.

Bibliography - Chapter IV

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**TITLE:** A Survey of the Theory of the Boundedness, Stability, and Asymptotic Behavior of Solutions of Linear and Non-Linear Differential and Difference Equations

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**ABSTRACT:**

A survey was made of the theory of the boundedness, stability, and asymptotic behavior of solutions of linear and nonlinear differential and difference equations. Boundedness and stability are qualitative properties, while asymptotic behavior is quantitative; all three are closely interrelated, and it is not easy to separate results into categories pertaining to one or the other property. The restriction to real differential equations has materially limited the scope of the results concerning asymptotic behavior.

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