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FOR THE RIEMANN PROBLEM

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ABSTRACT

Godunov's method and several other methods for computing solutions to the equations of gas dynamics use Riemann solvers to resolve discontinuities at the interface between cells. A new method is proposed here for solving the Riemann problem based on a global existence proof for the solution to the Riemann problem. The method is found to be very reliable and computationally efficient.

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1. INTRODUCTION

The introduction of a finite-difference scheme based on Riemann solvers by Godunov permitted the development of robust numerical methods for problems involving strong shocks. He proposed to consider all quantities in a computational cell as given by cell averages and to resolve the resulting discontinuities at the cell edges by the solution to a Riemann problem. The method has the important advantage that it automatically handles strong shocks and interactions and is able to predict cavitation should it occur.

The use of Godunov's method has attracted particular interest in the field of aerodynamics. Here one expects to encounter only ideal gas (polytropic) equations of state. A Riemann solver for the equations of gas dynamics was first proposed by Godunov. It was subsequently modified and improved by Chorin [2].

In this paper we take advantage of a global existence proof for the Riemann problem to propose an altogether different method for solving the Riemann problem. This method has the advantage that it is computationally very efficient and is guaranteed to converge to the solution of the Riemann problem.

2. THE RIEMANN PROBLEM IN GAS DYNAMICS

The gas dynamics equations in Eulerian coordinates are

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

$$e_t + \left((e + p)u\right)_x = 0$$

where ρ is the density of the gas, u is the velocity, ρu is the momentum, e is the energy per unit volume, and p is the pressure.

We assume that the gas is ideal and polytropic so that the equation of state is given by

$$p = \rho RT$$

where p is the pressure, ρ the density, T the temperature, and R the universal gas constant. Further, we assume that $\gamma > 1$, where γ is the ratio of specific heats.

We seek a solution to the gas dynamics equations with initial data

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = \begin{bmatrix} \rho_L \\ u_L \\ p_L \end{bmatrix} \quad \text{for} \quad x < 0$$

and

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = \begin{bmatrix} \rho_R \\ u_R \\ p_R \end{bmatrix} \quad \text{for} \quad x > 0.$$

The solution to the Riemann problem is illustrated in Figure 1 on a $x - t$ graph.

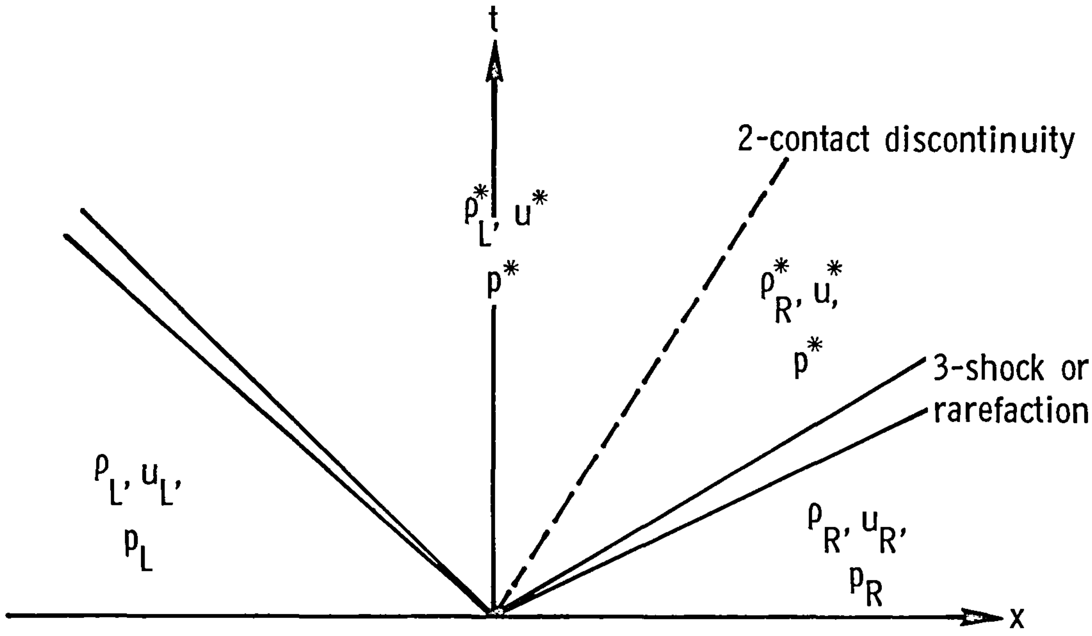

Figure 1: Solution to the Riemann problem.

The initial discontinuity is resolved by a system of waves consisting of a 1-shock or rarefaction, a 2-contact discontinuity, and a 3-shock or rarefaction. If cavitation is excluded the pressure and velocity on either side of the contact discontinuity are equal and are given by p^* and u^*.

3. THE GLOBAL EXISTENCE THEOREM

A parametric representation of the solution to the Riemann problem and global existence proof are discussed in Smoller [3]. We outline the discussion to relate it to our numerical scheme. Let

$$\beta = \frac{\gamma + 1}{\gamma - 1}$$

and

$$\tau = \frac{\gamma - 1}{2\gamma} .$$

We denote the sound speed by c, where $c = (\frac{\gamma p}{\rho})^{1/2}$. Thus c_L is the sound speed for the state (ρ_L, u_L, p_L). The solution to the Riemann problem is obtained by the following three one-parameter family of curves.

1 - family, for $x_1 \in \mathbb{R}$,

$$\frac{p^*}{p_L} = e^{-x_1}$$

$$\frac{\rho_L^*}{\rho_L} = f_1(x_1) \equiv \begin{cases} e^{-x_1/\gamma}, & x_1 \geq 0 \\ \dfrac{\beta + e^{x_1}}{1 + \beta e^{x_1}}, & x_1 \leq 0 \end{cases}$$

$$\frac{u^* - u_L}{c_L} = h_1(x_1) \equiv \begin{cases} \dfrac{2}{\gamma - 1}\left(1 - e^{-\tau x_1}\right), & x_1 \geq 0 \\ \dfrac{2(\tau)^{1/2}}{\gamma - 1} \dfrac{1 - e^{-x_1}}{\left(1 + \beta e^{-x_1}\right)^{1/2}}, & x_1 \leq 0 \end{cases}$$

2 - family, for $x_2 \in \mathbb{R}$

$$\frac{\rho_R^*}{\rho_L^*} = e^{x_2}$$

3 - family, for $x_3 \in \mathbb{R}$

$$\frac{p_R}{p^*} = e^{x_3}$$

$$\frac{\rho_R}{\rho_R^*} = f_3(x_3) \equiv \frac{1}{f_1(x_3)}$$

$$\frac{u_R - u^*}{c_R^*} = h_3(x_3) \equiv \begin{cases} \dfrac{2}{\gamma - 1}(e^{\tau x_3} - 1), & x_3 \geq 0 \\ \dfrac{2(\tau)^{1/2}}{\gamma - 1} \dfrac{(e^{x_3} - 1)}{(1 + \beta e^{x_3})^{1/2}}, & x_3 \leq 0. \end{cases}$$

To obtain a solution of the Riemann problem, we need to solve the above system of equations for the parameters x_1, x_2, and x_3. It should be noted that $x_1 < 0$ corresponds to a 1-shock and $x_1 > 0$ to a 1-rarefaction wave. A similar statement holds for the parameter x_3. Using the above representation the following theorem can be proved.

Theorem: Let (ρ_L, u_L, p_L) and (ρ_R, u_R, p_R) be any two states (not necessarily close). Then there is a unique solution to the Riemann problem with these initial states if and only if

$$u_R - u_L < \frac{2}{\gamma - 1}(c_L + c_R).$$

If this condition is violated, then a vacuum is present in the solution.

4. THE NUMERICAL METHOD

We first check whether

$$u_R - u_L < \frac{2}{\gamma - 1}(c_L + c_R).$$

If the condition is violated, then cavitation occurs.

To obtain the values x_1, x_2, and x_3 explicitly, we define

$$A = \frac{\rho_R}{\rho_L}$$

$$B = \frac{p_R}{p_L}$$

$$C = \frac{u_R - u_L}{c_L}.$$

We first solve for x_1, where x_1 satisfies the equation:

$$g(x_1) \equiv h_1(x_1) + \left(\frac{B}{A}\right)^{1/2} h_1(x_1 + \log B) - C = 0.$$

It should be noted that the function g is a monotone function. Hence, the equation can be solved efficiently by a numerical routine. We first locate an interval $[a,b]$ such that $g(a) < 0$ and $g(b) > 0$. With a and b as our initial guesses, a few iterations of the regula-falsi method are needed to obtain an accurate value for x_1.

We then have

$$x_3 = x_1 + \log B.$$

Finally, we obtain x_2 from the equation:

$$f_1(x_1)e^{x_2} f_3(x_3) = A.$$

Having obtained x_1, x_2, and x_3 we substitute them into the parametric representaion for the solution to the Riemann problem and obtain the values u^*, p^*, ρ_L^*, and ρ_R^*.

5. NUMERICAL RESULTS

We found that over a range of γ and on a variety of shock tube problems our Riemann solver converged to the exact solution within about five iterations of the regula-falsi method.

6. CONCLUSIONS

A numerical method for solving the Riemann problem in gas dynamics has been proposed. The method is based on an existence proof for the solution to the Riemann problem. Its advantages are that it can predict the occurrence of cavitation and it is computationally very efficient.

APPENDIX

Computer Program for the Riemann Solver

```
      subroutine riemann
      common/input/ul,ur,rhol,rhor,pl,pr
      common/output/pstar,ustar
      common/parm/gamma,beta,tau,a,b,c
c     input:ul,ur--velocities to the left and right of interface
c           rhol,rhor--densities
c           pl,pr--pressures
c     output:pstar--interface pressure
c           ustar--interface velocity
      beta=(gamma+1.)/(gamma-1.)
      tau=(gamma-1.)/(2.*gamma)
      cl=sqrt(gamma*pl/rhol)
      cr=sqrt(gamma*pr/rhor)
c     check for cavitaiton
      if((ur-ul).gt.(2.*(cl+cr)/(gamma-1.)))go to 3
      a=rhor/rhol
      b=pr/pl
      c=(ur-ul)/cl
      x=0.
c     locate interval on which g(x) has a zero
    9 if((g(x).gt.0.).or.(g(x+1.).lt.0.))then
            if(g(x).gt.0.)then
                x=x-1.
            else
                x=x+1.
            end if
      end if
      if((g(x).gt.0.).or.(g(x+1.).lt.0.))go to 9
c     do regula-falsi iterations
      xmax=x+1.
      xmin=x
      n=1
    8 x=xmax-g(xmax)*(xmax-xmin)/(g(xmax)-g(xmin))
            if(g(x).lt.0.) then
                xmin=x
            else
                xmax=x
            end if
      n=n+1
      if((abs(xmax-xmin).gt.0.0001).and.(n.le.10)) go to 8
      xl=xmax
c     calculate ustar,pstar
      pstar=exp(-1.*xl)*pl
      ustar=ul+cl*hl(xl)
      return
```

```
3 print *,'cavitation occurs'
  return
  end

  function hl(x)
  common/parm/gamma,beta,tau,a,b,c
  if(x.gt.0.)then
        hl=(2./(gamma-1.))*(1.-exp(-1.*tau*x))
  else
        hl=2.*sqrt(tau)*(1.-exp(-1.*x))/(gamma-1.)
        hl=hl/sqrt(1.+beta*exp(-1.*x))
  end if
  return
  end

  function g(x)
  external hl
  common/parm/gamma,beta,tau,a,b,c
  y=x+alog(b)
  g=hl(x)+sqrt(b/a)*hl(y)-c
  return
  end
```

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